Therefor, $K(5) = 25-25+6 = 6$

Now particular solution of the given equation is $u_x = \frac{5^x}{K(5)} = \frac{5^x}{6}$.

and the general solution of the corresponding homogeneous D.E. is

$$u_x = c_1 2^x + c_2 3^x$$

Hence the general solution of the given D.E. is

$$u_x = c_1 2^x + c_2 3^x + 5^x/6$$

Exercise-3 Solve the D.E.

$$f(x+3) + f(x+2) - 8f(x+1) - 12f(x) = 2x^2+5$$

Solution – We can write the give D.E. as

$$(E^3+E^2-8E-12)f(x) = 2x^2+5$$

$$\Rightarrow (E-3)(E+2)^2 f(x) = 2x^2+5$$

the particular solution of this D.E. is

$$f(x) = [(E-3)(E+2)^2]^{-1}(2x^2+5)$$

$$= [(\Delta-2)(\Delta+3)^2]^{-1}(2x^2+5)$$

$$= [\Delta^3+4\Delta^2-3\Delta-18]^{-1}(2x^2+5)$$

$$= -\frac{1}{18}\left[1+\frac{3\Delta-4\Delta^2-\Delta^3}{18}\right]^{-1}(2x^2+5)$$

$$= \frac{-1}{18}\left[1-\frac{3\Delta-4\Delta^2-\Delta^3}{18}+\frac{(3\Delta-4\Delta^2-\Delta^3)^2}{18^2}+\dots\right](2x^2+5)$$

$$= -\frac{1}{18}\left[1-\frac{\Delta}{6}+\frac{\Delta^2}{4}\right](2x^2+5),$$ the other terms will be zero as degree of $Q(x)$ is 2.

$$= -\frac{1}{18}\left[2x^2+5-\frac{4x+2}{6}+1\right]$$

$$= -\frac{1}{18}\left[2x^2-\frac{2}{3}x+\frac{17}{3}\right]$$

$$= -\frac{x^2}{9}+\frac{x}{27}-\frac{17}{54}$$

The general solution of the corresponding homogeneous D.E will be

$$f(x) = C_1 3^x + (C_2 + C_3 x)(-2)^x$$

Hence the complete general solution of the given non-homogeneous D.E will be

$$f(x) = C_1 3^x + (C_2 + C_3 x)(-2)^x - \frac{x^2}{9} + \frac{x}{27} - \frac{17}{54}$$

Exercise-4 Solve the following non-homogeneous linear D.E.

$$u_{x+2} + u_{x+1} + u_x = x2^x$$

Solution — The given D.E can be written as

$$(E^2 + E + 1) u_x = x 2^x$$

The particular solution of this equation can be obtained from result (6), taking $B = 2$ and $V(x) = x$. So

$$u_x = 2^x [(2+2\Delta)^2 + (2+2\Delta) + 1]^{-1} x$$

$$= 2^x [7 + 10\Delta + 4\Delta^2]^{-1} x$$

$$= \frac{2^x}{7} \left[1 + \frac{10\Delta + 4\Delta^2}{7}\right]^{-1} x$$

$$= \frac{2^x}{7} \left[1 - \frac{10\Delta + 4\Delta^2}{7}\right] x$$

$$= \frac{2^x}{7} \left[1 - \frac{10}{7}\Delta\right] x = \frac{2^x}{7}\left[x - \frac{10}{7}\right]$$

The general solution of corresponding homogeneous D.E will be

$$u_x = \lambda \left[\cos\left(\frac{\pi}{4} x + D\right)\right]$$

Hence the complete solution of the given D.E. is

$$u_x = \lambda \left[\cos\left(\frac{\pi}{4} x + D\right)\right] + \frac{2^x}{7}\left(x - \frac{10}{7}\right)$$

Exercise-5 Solve the following difference equations

(a) $y_{k+2} - 2y_{k+1} + y_k = 3k + 5$

(b) $y_{k+2} - 2y_{k+1} - 2y_k = k^2$

(c) $y_{k+2} - 5y_{k+1} + 6y_k = k^2 - 1$

7. Trial Method for finding the Particular Solution of a Non-Homogeneous Linear D.E.

This method is also known as the method of undetermined coefficients and is useful in finding the particular solution of a given non-homogeneous D.E when the earlier discussed operator method fails. Here corresponding to each term which is present in $Q(x)$, we consider a trial solution containing a number of different unknown constants which are to be determined by substitution of the trial solution in the given difference equation. Then by comparing the coefficients of the functions of like terms on both the sides we will have some algebraic equations containing the unknown constants. Finally on solving these algebraic equations, the values of unknown constants can be determined. The trial solution to be used in each case are shown in the following table where $A, B, A_0, A_1, \dots$ represent the unknown constants to be determined.

$Q(x)$	Trial Solution y_x
b (Constant)	A (Constant)
x or bx	$A_0 + A_1 x$
b^x or $k b^x$	$A b^x$
$\sin bx$ or $\cos bx$	$A \sin bx + B \cos bx$
x^n or $b x^n$	$A_0 + A_1 x + \dots + A_n x^n$
$b^x x^n$	$b^x (A_0 + A_1 x + \dots + A_n x^n)$
$a^x \sin x$ or $a^x \cos x$	$a^x (A \sin x + B \cos x)$
$a^x \sin bx$ or $a^x \cos bx$	$a^x (A \sin bx + B \cos bx)$

Remark – If $Q(x)$ is a linear combination of two or more terms given in the columns of $Q(x)$ then the trial solution is to be taken as the sum of the corresponding trial functions, with different unknown constant coefficients whose values are to be determined.

Exercise – 1 Solve the following D.E. with trial method.
$$y_{x+2} - 3y_{x+1} + 2y_x = 1$$

Solution – The auxiliary equation of given D.E. is
$$E^2 - 3E + 2 = 0$$
$$\Rightarrow (E-1)(E-2) = 0 \Rightarrow E = 1, 2$$
The general solution of the corresponding homogeneous D.E. is
$$y_x = C_1 + C_2 2^x$$
To find the particular solution of given non-homogeneous D.E we assume the trial solution
$$y_x = A \quad (\text{constant})$$
But C_1 (constant) already has been taken as a solution of homogeneous D.E. So, we can't take $y_x = A$ as a solution of given D.E. and try
$$y_x = Ax$$
Now, substituting $y_x = Ax$ in the given D.E. we get
$$A(x+2) - 3A(x+1) + 2Ax = 1$$
$$\Rightarrow -A = 1 \Rightarrow A = -1$$
Thus the required solution of given D.E is
$$y_x = C_1 + C_2 2^x - x$$

Exercise – 2 Solve the following D.E. with trial method.
$$y_{x+2} - 3y_{x+1} + 2y_x = 2^x$$

Solution – Here we observe that the auxiliary equation of given D.E is same as given in question (1). So, the general solution of the corresponding homogeneous D.E. is

$$y_x = C_1 + C_2 2^x$$

Now to find particular solution of given D.E. we consider the trial solution as per the given table

$$y_x = A2^x$$

But such a solution already has been taken of the homogeneous D.E. So, we can not take $y_x = A2^x$ and multiply it by x. and try

$$y_x = Ax2^x$$

Now, we substitute it in the given D.E. and get

$$A(x+2)2^{x+2} - 3A(x+1)2^{x+1} + 2Ax2^x = 2^x$$

$$\Rightarrow 4A(x+2)2^x - 6A(x+1)2^x + 2Ax2^x = 2^x$$

$$\Rightarrow 4A(x+2) - 6A(x+1) + 2Ax = 1$$

Comparing the constant terms on both the sides we get

$$8A - 6A = 1$$

$$\Rightarrow 2A = 1 \Rightarrow A = 1/2$$

Hence the complete solution of given D.E. is

$$y_x = C_1 + C_2 2^x + \frac{1}{2}x2^x$$

$$= C_1 + C_2 2^x + x2^{x-1}$$

Exercise-3 Solve the following D.E by operator method and find the values of b for which this method fails. Also apply the trial method for these values of b.

$$y_{k+2} - 3y_{k+1} + 2y_k = b^k$$

where b is a some constant.

Solution - We observe here that the auxiliary equation is same as in case of problem (1). So, the general solution of the corresponding homogeneous D.E is

$$y_k = C_1 + C_2 2^k$$

For the particular solution of given D.E. we observe that Q(x) is of the form AB^x with $A=1$ and $B=b$. So, particular solution is

$$y_k = \frac{b^k}{b^2-3b+2} = \frac{b^k}{(b-1)(b-2)} \; ; \; b \neq 1,2$$

Hence the required solution of given D.E is

$$y_k = C_1 + C_2 2^k + \frac{b^k}{(b-1)(b-2)} \; ; \; b \neq 1,2$$

Thus for $b=1$ and 2 the operator method is failed and we have obtained the solutions for $b=1$ and 2 by trial method in exercise 1 and 2 respectively.

Exercise-4 Solve the following D.E. by trial method.

$$y_{k+2} - 3y_{k+1} + 2y_k = 3^k$$

Solution - Try Your-self. (Ans. $y_k = C_1 + C_2 2^k + \frac{1}{2} 3^k$)

Exercise-5 Solve the following D.E. by using trial method:

$$y_{k+2} - 5y_{k+1} + 6y_k = 2$$

Satisfying the initial conditions $y_0 = 1$ and $y_1 = -1$

Solve— The auxiliary equation of the corresponding homogeneous D.E. is

$$E^2 - 5E + 6 = 0 \Rightarrow (E-2)(E-3) = 0$$
$$\Rightarrow E = 2, 3$$

So, the general solution of the corresponding homogeneous D.E is

$$y_k = c_1 2^k + c_2 3^k$$

To find particular solution, we assume trial f^n

$$y_k = A$$

So that the given D.E. is

$$A - 5A + 6A = 2 \Rightarrow A = 1$$

Hence the general solution of given D.E. is

$$y_k = c_1 2^k + c_2 3^k + 1$$

Now $y_0 = 1 \Rightarrow \quad c_1 + c_2 + 1 = 1 \Rightarrow c_1 + c_2 = 0 \quad$ (i)

$y_1 = -1 \Rightarrow \quad 2c_1 + 3c_2 + 1 = -1 \Rightarrow 2c_1 + 3c_2 = -2 \quad$ (ii)

Solving (i) and (ii), we get $c_1 = 2$ and $c_2 = -2$.

Therefore particular solution of given D.E. is

$$y_k = 2 \cdot 2^k - 2\, 3^k + 1$$
$$= 2^{k+1} - 2 \cdot 3^k + 1$$

To verify it, we put this solution in the given D.E. and its l.h.s. is

$$(2^{k+3} - 2 \cdot 3^{k+2} + 1) - 5(2^{k+2} - 2 \cdot 3^{k+1} + 1) + 6(2^{k+1} - 2 \cdot 3^k + 1)$$
$$= (4 - 10 + 6)2^{k+1} - (18 - 30 + 12)3^k + (7 - 5)$$
$$= (7-5) = 2 = \text{r.h.s.}$$

Exercise-6 By using trial method, find the solution of the D.E.

$$y_{k+2} - y_{k+1} - 2y_k = k^2$$

Solution – The given D.E can be written as

$$(E^2 - E - 2)y_k = k^2$$

The auxiliary equation of corresponding homogeneous D.E is.

$$E^2 - E - 2 = 0 \Rightarrow (E+1)(E-2) = 0$$
$$\Rightarrow E = -1, 2$$

So, the general solution of the corresponding homogeneous D.E. is $\quad y_k = c_1(-1)^k + c_2 2^k$

To find the particular solution of given D.E we consider the trial function

$$y_k = A_0 + A_1 k + A_2 k^2 \quad \text{(Since } k^2 \text{ is a polynomial of degree 2)}$$

Substituting this in the given D.E. we get

$$A_0 + A_1(k+2) + A_2(k+2)^2 - [A_0 + A_1(k+1) + A_2(k+1)^2]$$
$$-2[A_0 + A_1 k + A_2 k^2] = k^2$$

or $(-2A_0 + A_1 + 3A_2) + (-2A_1 + 2A_2)k - 2A_2 k^2 = k^2$

On comparing the coefficients of like powers of k, we get

$$-2A_0 + A_1 + 3A_2 = 0 \quad \text{—— (i)}$$
$$-2A_1 + 2A_2 = 0 \quad \text{—— (ii)}$$
$$-2A_2 = 1 \quad \text{—— (iii)}$$

From (i) to (iii), we get

$$A_0 = -1, \; A_1 = -\tfrac{1}{2} \text{ and } A_2 = -\tfrac{1}{2}$$

Therefore the particular solution of given equation is

$$y_k = -1 - \frac{k}{2} - \frac{k^2}{2}$$

Hence the complete general solution of given equation is

$$y_k = c_1(-1)^k + c_2 2^k - 1 - \frac{k}{2} - \frac{k^2}{2}$$

Exercise-7 Solve the following D.E. by using trial method.

$$y_{x+2} - 4y_{x+1} + 4y_x = 3x + 2^x$$

Solution – (Home exercise)

Hint – The general solution of the corresponding homogeneous D.E. is

$$y_x = (C_1 + C_2 x)2^x$$

To find particular solution we consider the trial function

$$y_x = A_0 + A_1 x + A x^2 2^x$$

$$\left\{ \text{Ans:- } (C_1 + C_2 x)2^x + 6 + 3x + \frac{1}{8}x^2 2^x \right\}$$

Exercise-8 Solve the following D.E. by using the trial method.

$$y_{h+1} - 2y_h = h+1$$

Solution – The general solution of the corresponding homogeneous D.E is

$$y_h = C_1 2^h$$

To find particular solution, we consider the trial function $y_h = A_0 + A_1 h$ (As $h+1$ is a linear fn)

Substituting it in the given D.E., we have

$$A_0 + A(h+1) - 2(A_0 + A_1 h) = h+1$$

or $-A_0 + A_1 - A_1 h = h+1$

comparing the terms on both the side we get

$$-A_1 = 1 \text{ and } -A_0 + A_1 = 1$$

$$A_0 = -2 \text{ and } A_1 = -1$$

$\therefore$ So, particular solution is $y_h = -2 - h$

Hence the general solution of given D.E is

$$y_h = C_1 2^h - h - 2$$

Exercise-8 Find the solution of the D.E.

$$y_{k+2} + y_{k+1} - 12\, y_k = 3^k + 10$$

Solution — The given D.E can be written as

$$(E^2 + E - 12)\, y_k = 3^k + 10$$

The auxiliary equation of Homogeneous DE is

$$E^2 + E - 12 = 0 \Rightarrow E = 3, -4$$

So, the general solution of homogeneous D.E is

$$y_k = c_1 3^k + c_2 (-4)^k$$

Now to find particular solution of the given DE we assume trial function

$$y_k = A_0 k\, 3^k + A$$

Putting this in given D.E. we get

$$A_0 (k+2)\, 3^{k+2} + A + A_0 (k+1)\, 3^{k+1} + A - 12\,(A_0 k\, 3^k + A) = 3^k + 10$$

Comparing the coefficients of 3^k and constant terms on both the sides —

$$-10A = 10 \quad \text{——— (i)}$$

$$\& \quad 9A_0 (k+2) + 3A_0 (k+1) - 12 A_0 k = 1 \quad \text{——— (ii)}$$

From (i) & (ii) we get

$$A_0 = \frac{1}{21} \text{ and } A = -1$$

Thus particular solution of DE is

$$y_k = \frac{k\,3^k}{21} - 1$$

Hence complete general solution of given D.E. is

$$y_k = c_1 3^k + c_2 (-4)^k + \frac{k\,3^k}{21} - 1$$

Exercise-9 Solve the following D.E by using the trial method.

(a) $2y_{k+1} - 5y_k = 3k+1$

(b) $y_{k+1} + 5y_k = 2k$

(c) $y_{k+1} - y_k = k$

(d) $y_{k+3} - 5y_{k+2} + 8y_{k+1} - 4y_k = k2^k$

(e) $y_{k+2} - 2y_{k+1} + y_k = 5+3k$

(f) $y_{x+2} - 3y_{x+1} + 2y_x = 1$; $y_0 = 1, y_1 = -1$

(g) $8y_{x+2} - 6y_{x+1} + y_x = 2^x$

(h) $(E^2 - 5E + 6)y(x) = x^2$; $y(0) = 1, y(1) = -1$

(i) $y_{x+2} - 5y_{x+1} + 6y_x = 4^x(x^2 - x + 5)$

8. **Solution of the Equation $y_{k+1} = Ay_k + C$ $(A \neq 0)$**

The equation is first order non-homogeneous D.E with constant coefficient. Here A and C are certain constants. Consider the D.E.

$$y_{k+1} = Ay_k + C \; ; \; k = 0, 1, 2, \ldots \qquad (1)$$

Putting $k=0$ in (1) we get

$$y_1 = Ay_0 + C \qquad (2)$$

Putting $k=1$ in (1), we get

$$y_2 = Ay_1 + C$$
$$= A(Ay_0 + C) + C, \text{ using } (2)$$
$$= A^2 y_0 + C(1+A) \qquad (3)$$

Again putting $k=2$, in (1) we get

$$y_3 = Ay_2 + C$$

$$= A[A^2 y_0 + C(1+A)] + C$$
$$= A^3 y_0 + AC(1+A) + C$$
$$= A^3 y_0 + C(1+A+A^2).$$

In general, we get

$$y_k = A^k y_0 + C(1+A+A^2+\cdots+A^{k-1})$$
$$= \begin{cases} A^k y_0 + \dfrac{C(1-A^k)}{1-A} & ; \text{ if } A \neq 1 \\ y_0 + Ck & ; \text{ if } A = 1 \end{cases}$$

Exercise-1 Find the solution of

$$y_{k+1} = 2y_k - 1 \quad ; \quad k = 0, 1, 2,$$

with initial condition $y_0 = 5$

Solution – Comparing the given D.E with

$$y_{k+1} = Ay_k + C$$

we get $A = 2$ and $C = -1$, therefore

$$y_k = A^k y_0 + C(1-A^k)/(1-A)$$
$$= 2^k.5 - 1(1-2^k)/(1-2)$$
$$= 5.2^k + 1 - 2^k = 2^k(5-1)+1$$
$$= 2^{k+2}+1$$

It gives the series

$$y_0, y_1, y_2, y_3, y_4, \ldots$$
$$= 5, 9, 17, 33, 65, 129,$$

Exercise-2 Solve the D.E

$$y_{k+1} = -y_k + 2 \quad ; \quad k = 0, 1, 2,$$

and write the complete series.

Solution – Here $A = -1$ and $C = 2$, therefore the solution is given by

$$y_k = A^k y_0 + \frac{C(1-A^k)}{1-A}$$
$$= (-1)^k y_0 + \frac{2[1-(-1)^k]}{1+1}$$
$$= (-1)^k y_0 + 1 - (-1)^k$$
$$= (-1)^k (y_0 - 1) + 1 \quad ; \quad k = 0, 1, 2, \ldots$$

If $k=0$ or an even integer, then $(-1)^k=1$ and if k is an odd integer then $(-1)^k=-1$. Therefore we have the following sequence of values

$$y_0, -y_0+2, y_0, -y_0+2, \ldots$$

Exercise-3 Solve the D.E.

$$2y_{k+1}-y_k=4 \; ; \; k=0,1,2,\ldots$$

with initial condition $y_0=3$

Solution- The given equation can be written as

$$y_{k+1}=\frac{1}{2}y_k+2 \; ; \; k=0,1,2,\ldots$$

Here $A=\frac{1}{2}$ and $C=2$

Therefore the solution is given by

$$y_k = A^k y_0 + C\cdot\frac{1-A^k}{1-A}$$

$$= 3\left(\tfrac{1}{2}\right)^k + 2\cdot\frac{1-(1/2)^k}{1-1/2}$$

$$= 3\left(\tfrac{1}{2}\right)^k + 4\left[1-(1/2)^k\right]$$

$$= 4-(1/2)^k, \; k=0,1,2,\ldots$$

Thus we obtain the following sequence of values

$$3, 7/2, 15/4, 31/8, \ldots$$

Exercise-4 Solve the D.E

$$2y_{k+1}-y_k=2$$

Solution- The given D.E. can be written as

$$y_{k+1}=\frac{1}{2}y_k+1$$

Here $A=1/2$ & $C=1$

The solution of the given D.E. is

$$y_k = y_0 A^k + \frac{C(1-A^k)}{1-A}$$

$$= y_0(1/2)^k + \left[1-(1/2)^k\right]/\{1-(1/2)\}$$

$$= (y_0-2)(1/2)^k+2 = 2^k y_0 - 2(2^k-1)$$

The sequence is

$$y_0, 2y_0-2, 4y_0-6, \ldots$$

Alternatively, we can solve the above D.E. by trial method as follows—

The general solution of the corresponding homogeneous D.E. is

$$y_k = c_1 (1/2)^k$$

To find particular solution, we try the trial f^n

$$y_k = A$$

Substituting this in the given D.E. we get

$$2A - A = 2 \Rightarrow A = 2$$

Thus the general solution of the given D.E. is

$$y_k = c_1 (1/2)^k + 2$$

Putting $k=0$ on both sides

$$y_0 = c_1 + 2 \Rightarrow c_1 = y_0 - 2$$

Therefore $$y_k = (y_0 - 2)(1/2)^k + 2$$

9. Some Special Forms of D.Es.—

Exercise-1 Solve the D.E.

$$(E-a)(E-b)\, y_k = 0$$

Solution— If we take $(E-b)y_k = U_k$, then the given D.E. will be

$$(E-a)\, U_k = 0$$

The solution of this D.E. is

$$U_k = c_1 a^k$$

$$\Rightarrow \quad (E-b)\, y_k = c_1 a^k \Rightarrow y_{k+1} - b y_k = c_1 a^k$$

The general solution of the corresponding homogeneous D.E. is $y_k = c_2 b^k$

To find particular solution we consider the trial solution $y_k = A_0 a^k$

Now substituting this trial solution in given D.E. we get

$$A_0 a^{k+1} - b A_0 a^k = c_1 a^k$$

$$\Rightarrow a A_0 - b A_0 = c_1 \Rightarrow A_0 = \frac{c_1}{a-b}$$

Therefore the particular solution is

$$y_k = \frac{c_1}{a-b} a^k$$

Hence the complete solution of the given D.E. is

$$y_k = c_2 b^k + \frac{c_1}{a-b} a^k$$

or $\quad y_k = c_2 b^k + c_3 a^k$, where $c_3 = \dfrac{c_1}{a-b}$

Exercise-2 Solve the D.E.

$$y_k \cdot y_{k+2} = y_{k+1}^2$$

Solution — The given D.E. can be written as

$$y_k \cdot y_{k+2} - y_{k+1}^2 = 0$$

or $$\frac{y_k\, y_{k+2} - y_{k+1}^2}{y_k \cdot y_{k+1}} = 0$$

or $$\frac{y_{k+2}}{y_{k+1}} - \frac{y_{k+1}}{y_k} = 0$$

or $$\Delta\left(\frac{y_{k+1}}{y_k}\right) = 0$$

or $$\frac{y_{k+1}}{y_k} = c \text{ (constant)}$$

or $$y_{k+1} = c\, y_k$$

The solution of this D.E. is

$$y_k = c^k y_0$$

Exercise-3 Solve the D.E.

$$(u_x+5)u_{x+1}+u_x+9=0$$

Solution— Putting $(u_x+5)=\frac{v_{x+1}}{v_x}$ we get

$$\frac{v_{x+1}}{v_x}\left(\frac{v_{x+2}}{v_{x+1}}-5\right)+\left(\frac{v_{x+1}}{v_x}-5\right)+9=0$$

or $\quad \frac{v_{x+2}}{v_x}-4\frac{v_{x+1}}{v_x}+4=0$

or $\quad v_{x+2}-4v_{x+1}+4v_x=0 \qquad (1)$

which is a homogeneous D.E. of order 2 in v_x.
Solving it for v_x we get

$$v_x=(c_1+c_2x)2^x$$

Hence the general solution of given D.E. is

$$u_x=\frac{v_{x+1}}{v_x}-5=\frac{\{c_1+c_2(x+1)\}2^{x+1}}{\{c_1+c_2x\}2^x}-5$$

$$=\frac{\{c_1+c_2(x+1)\}2}{(c_1+c_2x)}-5$$

Exercise-4 Solve the D.E.

$$u_xu_{x+1}-au_x+b=0$$

Solution— We have the D.E

$$(u_{x+1}-a)u_x+b=0$$

Putting $u_x=\frac{v_{x+1}}{v_x}$, we get

$$\frac{v_{x+1}}{v_x}\cdot\frac{v_{x+2}}{v_{x+1}}-a\frac{v_{x+1}}{v_x}+b=0$$

or $\quad \frac{v_{x+2}}{v_x}-a\frac{v_{x+1}}{v_x}+b=0$

or $\quad v_{x+2}-av_{x+1}+bv_x=0 \qquad (1)$

The equation(1) is a homogeneous D.E of order two. The auxiliary equation of this equation is $E^2 - aE + b = 0$

The roots of auxiliary equation are

$$E = \frac{a \pm \sqrt{a^2-4b}}{2} = \frac{a}{2} \pm \frac{i\sqrt{4b-a^2}}{2}$$

Now to change these roots into polar form we put

$$\frac{a}{2} + \frac{i\sqrt{4b-a^2}}{2} = r(\cos\theta + i\sin\theta)$$

comparing real and imaginary parts we get

$$r\cos\theta = \frac{a}{2} \quad \text{and} \quad r\sin\theta = \frac{\sqrt{4b-a^2}}{2}$$

Now $r^2(\cos^2\theta + \sin^2\theta) = b \Rightarrow r = \sqrt{b}$

and $\tan\theta = \frac{\sqrt{4b-a^2}}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{4b-a^2}}{a}\right)$

Thus $V_x = \lambda(\sqrt{b})^x \cos(\theta x + D)$

and the solution of given D.E. is

$$u_x = \frac{\lambda(\sqrt{b})^{x+1}\cos\{\theta(x+1)+D\}}{\lambda(\sqrt{b})^x \cos(\theta x + D)}$$

$$= \frac{\sqrt{b}\cos\{\theta(x+1)+D\}}{\cos(\theta x + D)}$$

Exercise-5 Show that the general solution of the D.E $u_{x+1}u_x + a u_{x+1} + b u_x + c = 0$ can be written in the form

$$u_x = \frac{A\alpha^{x+1} + \beta^{x+1}}{A\alpha^x + \beta^x} - a$$

where α and β are the roots of the equation $x^2 - (a-b)x + (c-ab) = 0$ and A is arbitrary constant.

Solution — We have the D.E.

$$u_{x+1}(u_x+a)+bu_x+c=0$$

Putting $u_x+a=\frac{v_{x+1}}{v_x}$ we get

$$\left(\frac{v_{x+2}}{v_{x+1}}-a\right)\cdot\frac{v_{x+1}}{v_x}+b\left(\frac{v_{x+1}}{v_x}-a\right)+c=0$$

or $$\frac{v_{x+2}}{v_x}-(a-b)\cdot\frac{v_{x+1}}{v_x}+c-ab=0$$

or $$v_{x+2}-(a-b)v_{x+1}+(c-ab)v_x=0 \qquad \text{—(1)}$$

This is the second order homogeneous D.E. the auxiliary equation of this D.E is

$$E^2-(a-b)E+(c-ab)=0 \qquad \text{—(2)}$$

$$\Rightarrow (E-\alpha)(E-\beta)=0$$

where α and β are the roots of (2). So, the general solution of D.E.(1) is

$$v_x=c_1\alpha^x+c_2\beta^x$$

and the solution of given D.E. is

$$u_x=\frac{c_1\alpha^{x+1}+c_2\beta^{x+1}}{c_1\alpha^x+c_2\beta^x}-a$$

$$=\frac{\frac{c_1}{c_2}\alpha^{x+1}+\beta^{x+1}}{\frac{c_1}{c_2}\alpha^x+\beta^x}-a=\frac{A\alpha^{x+1}+\beta^{x+1}}{A\alpha^x+\beta^x}-a$$

x —— x