Measurement of Cyclic Variation :-

To find the cyclic components of a time-series, we use the method, generally called residual method. In this method 3 other components like trend, seasonal, irregular variations are removed from the time-series. But this method is not appropriate. So, we need some other methods.

(1) **Harmonic Analysis** :- Harmonic analysis is an application of Fourier forier series in time-series and is used to determine the cyclic variation. Fourier series Forier is a sum of sin's and cosin's f^ns. Thus, for a time-series U_t y_t with period of oscillation λ can be represented as

$$U_t \; y_t = a_0 + \left[a_1 \sin\left(\frac{2\pi}{\lambda}t\right) + a_2 \sin\left(\frac{2\pi}{\lambda}2t\right) + \cdots\right]$$

$$+ \left[b_1 \cos\left(\frac{2\pi}{\lambda}t\right) + b_2 \cos\left(\frac{2\pi}{\lambda}2t\right) + \cdots\right]$$

$$= a_0 + \sum_{i=1}^{\infty}\left[a_i \sin\left(\frac{2\pi}{\lambda}it\right) + b_i \cos\left(\frac{2\pi}{\lambda}it\right)\right]$$

where, $a_0 = \frac{1}{n}\sum_{t=1}^{n} y_t$, $a_i = \frac{2}{n}\sum_{t=1}^{n} y_t \sin\left(\frac{2\pi}{\lambda}it\right)$,

$$b_i = \frac{2}{n}\sum_{i=1}^{n} y_t \cos\left(\frac{2\pi}{\lambda}it\right), \quad i = 1, 2, \cdots$$

n be the no. of terms in time-series. After representing an oscillatory component of a time-series by harmonic or fourier analysis, we observe that the hidden periodicity in the time-series represented by λ which is unknown. If we can estimate the value of λ, then we are able to find the values of other involved parameters (i.e. we can estimate cyclic variation). To estimate the value of hidden periodicity λ, we may use the periodogram analysis.

Periodogram Analysis :- In periodogram analysis, we assume that the trend and seasonal components of a time-series has been eliminated. In other words, the time-series y_t consist only 2 components

(1) Periodic with period λ and amplitude a

(2) Random component ϵ_i, which is uncorrelated with any ~~cly~~ cyclic movement for a long time-series. Thus, we can write

$$y_t = a \sin\left(\frac{2\pi}{\lambda} t\right) + \epsilon_t \qquad \text{——(1)}$$

where,

$$\text{cov}(\varepsilon_i, \varepsilon_j) = 0$$

Now,

$$\text{Var}(\varepsilon_i) = \sigma^2$$

$$\text{cov}\left(\varepsilon_i, \sin\left(\frac{2\pi}{\lambda} t\right)\right) = 0$$

$$\text{cov}\left(\varepsilon_j, \cos\left(\frac{2\pi}{\lambda} t\right)\right) = 0$$

Now, let us define $A = \frac{2}{n} \sum_{t=1}^{n} y_t \sin\left(\frac{2\pi}{\lambda} t\right)$ and

$$B = \frac{2}{n} \sum_{t=1}^{n} y_t \cos\left(\frac{2\pi}{\lambda} t\right) \quad \text{——(2)}$$

~~where~~

μ is the some trial period further

define

$$\delta^2(\mu) = A^2 + B^2 \quad \text{——(3)}$$

$\delta(\mu)$ is the intensity corresponding to trial period μ. Now, using (1) and (2), we get

$$A = \frac{2}{n} \sum_{t=1}^{n} \left(a \sin \frac{2\pi t}{\lambda} + \varepsilon_i \right) \cos\left(2\pi\right.$$

Nov.	30·69	1·13	34·680
Dec.	31·14	1·36	42·350

in a raw / unprepared data

2·6. Measurement of Cyclic Movement. A crude method of measuring cyclic fluctuations (C_t) of a time series (U_t) is provided by the so-called *Residual Approach*, which essentially consists in eliminating the two components, *viz.*, trend (T_t) and seasonal variation (S_t) and then removing the random component or irregular variation (R_t) by averaging. Using the multiplicative model

$$U_t = T_t \cdot S_t \cdot R_t \cdot C_t,$$

the steps involved in the calculation of C_t are :

(*i*) Calculate trend value T_t by moving average method and seasonal index S_t (which is to be taken in fractional form and not in percentage form), preferably by a moving average method.

(*ii*) Divide U_t by $T_t \times S_t$.

(*iii*) The resulting value gives C_t, the cyclic component.

The use of moving average method of suitable period averages (smoothens) out the random component R_t. Since the method requires, for its success, a number of conditions to be satisfied, it is very seldom used.

Example 2·13. *Obtain the indices of cyclical variation for the data in example 2·9.*

Solution. Cyclical variations are obtained by eliminating trend and seasonal variations from the given data. The seasonal variations are eliminated by dividing every figure by seasonal index of the corresponding quarters, and subtracting 100 from the result so obtained.

CALCULATION OF CYCLICAL VARIATIONS

Year	(Trend Eliminated figures ÷ Seasonal Index) × 100 = y: I Qrt.	II Qrt.	III Qrt.	IV Qrt.	Cyclical Variations (y − 100): I Qrt.	II Qrt.	III Qrt.	IV Qrt.
1970	118·6	111·7	105·3	105·3	18·6	11·7	5·3	5·3
1971	93·6	104·3	107·6	102·6	−6·4	4·3	7·6	2·6
1972	84·5	90·6	92·0	89·7	−15·5	−9·4	−8·0	−10·3
1973	92·4	97·4	95·8	96·7	−7·6	−2·6	−4·2	−3·3
1974	115·2	99·7	103·3	109·7	15·2	−0·3	3·3	9·7

Harmonic Analysis provides a sophisticated method of determining he cyclic component of a time series.

Harmonic Analysis. From mathematical analysis we know that any unction, U_t under some very general conditions, can be represented by a *ourier series, viz.*, a series of sums of sine and cosine functions. Thus for -time series (U_t) with period of oscillation λ, we have

$$U_t = a_0 + a_1 \sin \frac{2\pi t}{\lambda} + a_2 \sin \frac{2\pi}{\lambda} . 2t + \dots$$

$$+ b_1 \cos \frac{2\pi}{\lambda} t + b_2 \cos \frac{2\pi}{\lambda} . 2t + \dots \qquad \dots(*)$$

vhere

$$a_i = \frac{2}{n} \sum_{t=1}^{n} U_t . \sin \left(\frac{2\pi}{\lambda} it \right), \ (i = 1, 2, \dots)$$

$$b_i = \frac{2}{n} \sum_{t=1}^{n} U_t . \cos \left(\frac{2\pi}{\lambda} . it \right), \ (i = 1, 2, \dots)$$

$$a_0 = \frac{1}{n} \sum_{t=1}^{n} U_t$$

here n is the number of terms in the time series. For instance, if the eriod of oscillation is 12 months and $U_1, U_2, \dots, U_{12}$ is the series or erage of series for a number of years, then the constants a_i's and b_i's re given by

$$a_0 = \frac{1}{12} \sum_{t=1}^{12} U_t$$

$$a_i = \frac{2}{12} \sum_{t=1}^{12} U_t \sin \left(\frac{2\pi}{12} it \right), \ (i = 1, 2, \dots, 6)$$

$$b_i = \frac{2}{12} \sum_{t=1}^{12} U_t . \cos \left(\frac{2\pi}{12} it \right), \ (i = 1, 2, \dots, 5)$$

excellent/fine/superior

So far λ in (*) is regarded as a known constant. *Periodogram Analysis* provides an elegant method of determining λ.

Let us consider a time series in which the trend and the seasonal component have been eliminated. In other words, the resulting series 'U_t' consists of only two components, one periodic with period λ, (say), and amplitude 'a' and the other, the random component ϵ_t (say) which is uncorrelated with any cyclic movement, for long time at least. Thus

great size or extent

$$U_t = a \sin \frac{2\pi}{\lambda} t + \epsilon_t \qquad \ldots(2\cdot42)$$

$$\left.\begin{array}{l} \text{Cov}\ (\epsilon_i, \epsilon_j) = 0,\ (i \neq j)\ ;\quad \text{Var}\ (\epsilon_i) = \sigma^2 \\ \text{Cov}\ \left(\epsilon_t, \sin \dfrac{2\pi t}{\lambda}\right) = 0\ :\ \text{Cov}\left(\epsilon_t, \cos \dfrac{2\pi t}{\lambda}\right) = 0 \end{array}\right\} \qquad \ldots(2\cdot43)$$

Let us consider

$$\left.\begin{array}{l} A = \dfrac{2}{n} \displaystyle\sum_{t=1}^{n} U_t \cos \dfrac{2\pi t}{\mu} \\ B = \dfrac{2}{n} \displaystyle\sum_{t=1}^{n} U_t \sin \dfrac{2\pi t}{\mu} \end{array}\right\} \qquad \ldots(2\cdot44)$$

where μ is arbitrary and define

$$S^2(\mu) = A^2 + B^2 \qquad \ldots(2\cdot45)$$

which is known as intensity corresponding to the trial period μ.

Substituting from (2·42) in (2·44) and using (2·43), we get

intensity - the quality of being intense

intense - of extreme force, degree or strength

$$A = \frac{2}{n} \sum_{t=1}^{n} \left(a \sin \frac{2\pi t}{\lambda} + \epsilon_t\right) \cos \frac{2\pi t}{\mu}$$

$$= \frac{a}{n} \sum_{t=1}^{n} 2 \sin \frac{2\pi t}{\lambda} \cos \frac{2\pi t}{\mu}$$

$$= \frac{a}{n} \sum_{t=1}^{n} 2 \sin \alpha t \cos \beta t,$$

where $\qquad \alpha = \dfrac{2\pi}{\lambda}$ and $\beta = \dfrac{2\pi}{\mu}$.

$$\therefore \qquad A = \frac{a}{n} \sum_{t=1}^{n} \Big[\sin (\alpha + \beta) t + \sin (\alpha - \beta) t \Big] \qquad \ldots(2\cdot46)$$

Let $\qquad S = \displaystyle\sum_{t=1}^{n} \sin (\alpha + \beta) t$

$2 \sin A \cos B = \sin (A+B) + \sin (A-B)$

$\sin A \sin B = \cos(A-B) - \cos(A+B)$
$\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}$

$$= S \cdot \sin\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{2}\sum_{t=1}^{n} 2 \sin (\alpha+\beta)t \cdot \sin\left(\frac{\alpha+\beta}{2}\right)$$

$$= \frac{1}{2}\sum_{t=1}^{n}\left[\cos\left\{(\alpha+\beta)t - \frac{\alpha+\beta}{2}\right\} - \cos\left\{(\alpha+\beta)t + \frac{\alpha+\beta}{2}\right\}\right]$$

$$= \frac{1}{2}\left[\left\{\cos\left(\frac{\alpha+\beta}{2}\right) - \cos\frac{3(\alpha+\beta)}{2}\right\} + \left\{\cos\frac{3(\alpha+\beta)}{2} - \cos\frac{5(\alpha+\beta)}{2}\right\} + \dots + \left\{\cos\frac{(2n-1)(\alpha+\beta)}{2} - \cos\frac{(2n+1)(\alpha+\beta)}{2}\right\}\right]$$

$$= \frac{1}{2}\left[\cos\frac{\alpha+\beta}{2} - \cos\frac{(2n+1)(\alpha+\beta)}{2}\right]$$

$$= \sin\left\{\frac{(n+1)(\alpha+\beta)}{2}\right\} \cdot \sin\frac{n(\alpha+\beta)}{2}.$$

$$\Rightarrow \quad S = \sum_{t=1}^{n} \sin(\alpha+\beta)t = \frac{\sin\frac{n(\alpha+\beta)}{2} \cdot \sin\frac{(n+1)(\alpha+\beta)}{2}}{\sin\left(\frac{\alpha+\beta}{2}\right)}$$

Similarly we can get the value of $\sum_{t=1}^{n} \sin(\alpha-\beta)t$.

Substituting in (2·46), we get

$$A = \frac{a}{n}\left[\frac{\sin\frac{n(\alpha+\beta)}{2}\sin\frac{(n+1)(\alpha+\beta)}{2}}{\sin\left(\frac{\alpha+\beta}{2}\right)} + \frac{\sin\frac{n(\alpha-\beta)}{2}\sin\frac{(n+1)(\alpha-\beta)}{2}}{\sin\left(\frac{\alpha-\beta}{2}\right)}\right]$$

If $\alpha \neq \beta$, then $A \to 0$ for a large n, since then the expression in bracket is bounded for all α, β and n. However if $\alpha \to \beta$, *i.e.*, $\alpha - \beta \to 0$, then for large n, we get

$$A = \lim_{n\to\infty} \frac{a}{n}\,[\text{some finite quantity}]$$

$$+ \lim_{n\to\infty} \frac{a}{n} \cdot \frac{\sin\frac{n(\alpha-\beta)}{2}\sin\frac{(n+1)(\alpha-\beta)}{2}}{\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$= a \sin\frac{(n+1)(\alpha-\beta)}{2} \text{ for large } n, \left(\text{since } \lim_{\theta\to 0}\frac{\sin n\theta}{\sin\theta} = n\right)$$

Thus for large n,

if $\alpha \neq \beta \Rightarrow \lambda \neq \mu$, then $A \to 0$ and

if $\alpha \to \beta \Rightarrow \lambda \to \mu$, then $A \to a \sin\frac{(n+1)(\alpha-\beta)}{2}$

Similarly it can be shown that for large n,

$$\text{if } \alpha \neq \beta \;\Rightarrow\; \lambda \neq \mu, \quad \text{then } B \to 0 \text{ and}$$

$$\text{if } \alpha \to \beta \;\Rightarrow\; \lambda \to \mu, \quad \text{then } B \to a \cos \frac{(n+1)(\alpha-\beta)}{2}$$

Thus if the arbitrary number 'μ' is exactly the period of oscillation (λ) of the series, then

$$S^2(\mu) = a^2 \;\Rightarrow\; S(\mu) = a \qquad \ldots(2\cdot47)$$

On the other hand if $\lambda \neq \mu$, then $S(\mu) = 0$. Thus $S(\mu)$ remains small unless the trial period μ approaches the true period λ in which case its value is equal to the amplitude 'a'. This conclusion forms the basis of Periodogram Analysis, which is summed up as follows :

From given time series $U_1, U_2, \ldots, U_n$ calculate A and B as defined in (2·44) for different values of μ from 0 to n and compute $S(\mu)$ using (2·45). The graph obtained on plotting $S(\mu)$ against μ is known as periodogram. The values of μ corresponding to the 'significant' maximum values of S provide the fundamental periods of oscillation, provided no μ, is a multiple of another μ ; the corresponding values of S provide the corresponding amplitudes.

Remark. The obvious drawback of Harmonic Analysis lies in 'huge calculations'. If by drawing the graph of the time series we can guess the true periods of oscillation, it may be necessary to compute S for only those values of the trial period μ which are in the neighbourhood of approximate (guessed) values.

EXERCISE 2·D

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