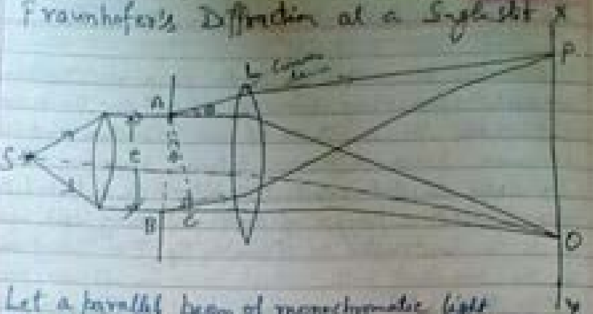
Fraunhofer's Diffraction at a Single Slit



Let a parallel beam of monochromatic light of wavelength λ be incident normally upon a narrow slit of width $AB = e$ placed $\perp$ to the plane of paper. Let the diffracted light be focussed by a convex lens L on a screen XY be placed on the focal plane of the lens. In terms of wave theory, a plane wavefront is incident normally on the slit AB. According to Huygen's principle, each point in AB sends out secondary wavelets in all directions. The rays proceeding in the same direction as the incident rays are focussed at O, while those diffracted through an angle θ are focussed at P. Let us find out the resultant intensity at P.

Let AC be the perpendicular drawn on BC. As the optical paths from the plane AC to P are equal, the path difference between the wavelets from A to B in the direction θ

$$BC = e \sin\theta$$

Now phase difference is $\phi = \frac{2\pi}{\lambda} \times BC$

$$= \frac{2\pi}{\lambda} \cdot e \sin\theta$$

Using the expression of the resultant wave in diffraction

$$R = \frac{a \sin \frac{n\delta}{2}}{\sin \frac{\delta}{2}}$$

Let the width AB of the slit be divided into n equal parts

Suppose n is infinitely large and a & d are infinitesimally small. Let $n\delta = 2\alpha$

$$R = \frac{a \sin \frac{\phi}{2}}{\sin \frac{\phi}{2n}}$$

$$R = \frac{a \sin \frac{2\pi}{\lambda} \times \frac{e \sin\theta}{2}}{\frac{2\pi}{\lambda} \times \frac{e \sin\theta}{2n}} = a \sin \left[\frac{\frac{\pi e \sin\theta}{\lambda}}{\frac{\pi e \sin\theta}{n\lambda}} \right]$$

$$\boxed{\alpha = \frac{\pi e \sin\theta}{\lambda}}$$

$$R = \frac{a \sin\alpha}{\sin\left(\frac{\alpha}{n}\right)}$$

as $n \to \infty$; $\frac{\alpha}{n} \to 0$ $\quad \sin \frac{\alpha}{n} \to \frac{\alpha}{n}$

$$R = a \frac{\sin\alpha}{\frac{\alpha}{n}} = \frac{an \sin\alpha}{\alpha}$$

Product na is still finite let it $na = A$

So $\boxed{R = A \frac{\sin\alpha}{\alpha}}$

$$R^2 = A^2 \frac{\sin^2\alpha}{\alpha^2}$$

$$I = \frac{A^2 \sin^2\alpha}{\alpha^2}$$

Condition of Maxima & Minima $\quad$ when $\theta \to 0°$

$\Rightarrow \alpha \to 0°$

$\lim_{\alpha \to 0} \frac{\sin\alpha}{\alpha} \to 1$

$I_{max} = A^2$

Condition of Minima

For $I_{min} = 0$

$\sin\alpha = 0$; but $\alpha \neq 0$

$\boxed{\alpha = n\pi}$ $n \neq 0$

Because $\alpha = 0$ is the point of Central Maxima

Therefore $\alpha = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots \ldots \pm n\pi$ are the point of minimum Intensity

* Condition of Secondary Maxima / The Directions of Maximum Intensity : Let us Differentiate I with respect to α and equate to zero i.e.

$$\frac{dI}{d\alpha} = 0$$

$$\frac{d}{d\alpha}\left[A^2\left(\frac{\sin\alpha}{\alpha}\right)^2\right] = 0$$

$$A^2\left(\frac{2\sin\alpha}{\alpha}\right) \cdot \frac{1}{\alpha} \cdot \cos\alpha - \frac{1}{\alpha^2}\sin\alpha = 0$$

$$A^2\left[\frac{2\sin\alpha}{\alpha}\right]\frac{1}{\alpha^2}\left[\alpha\cos\alpha - \sin\alpha\right] = 0$$

Secondary maxima

$$\Rightarrow \frac{\alpha\cos\alpha - \sin\alpha}{\alpha^2} = 0$$

When $\sin\alpha = 0$ or $\alpha = \pm n\pi$

$\alpha\cos\alpha - \sin\alpha = 0$ Position of maxima given by roots of the eqn $\alpha = \tan\alpha$

$$\Rightarrow \alpha\cos\alpha = \sin\alpha$$

$$\alpha = \tan\alpha \quad \text{------} \quad 1$$

Let us say $y = \alpha$

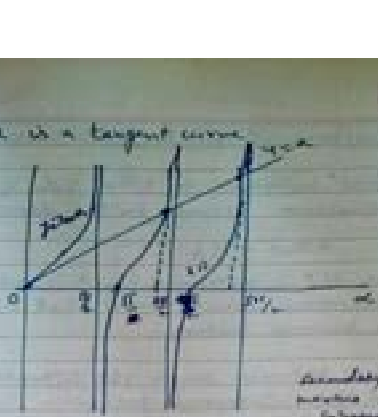
Then $y = \tan\alpha$

The secondary maximum intensity points can be obtained by solving two curves $y = \alpha$

$y = \tan\alpha$

$y = \alpha$ is a straight line passing through origin and are inclined at $45°$

$y = \tan\alpha$ is a tangent curve



Secondary maxima is obtained by intersection of $y = \alpha$ & $y = \tan\alpha$

$$\alpha = 0,\ \frac{3\pi}{2},\ \frac{5\pi}{2},\ \frac{7\pi}{2} \ldots\ldots$$

$$I_0 = A^2\left(\frac{\sin 0}{0}\right)^2 = A^2$$

First Subsidiary Maxima $\quad I_1 = A^2\left\{\frac{\sin 3\pi/2}{3\pi/2}\right\}^2 = \frac{A^2}{22}$

IInd Subsidiary Maxima $\quad I_2 = A^2\left\{\frac{\sin 5\pi/2}{5\pi/2}\right\}^2 = \frac{A^2}{61}$

and so on $\quad 1 : \frac{1}{22} : \frac{1}{61} : \frac{1}{121} :$

Most of the incident light is concentrated in the principal maximum which occurs in the direction given by

$$\alpha = 0$$

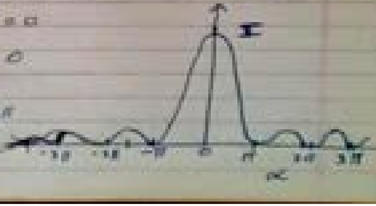
$$\frac{\pi e \sin\theta}{\lambda} = 0$$

$$\theta = 0$$

So we have alternatively maxima & Minima

Minima lie at $\alpha = \pm\pi, \pm 2\pi, \pm 3\pi$

Intensity distribution plot shown in fig



* What happens if slit is made narrower—

The 'first' minima on either side of the central maximum occurs in the direction θ is given by

$$e \sin\theta = \pm\lambda$$

When the slit is narrowed i.e. e is reduced the angle θ increases which means that the central maximum becomes wider.

* Difference between single slit diffraction pattern & double slit interference pattern—

The diffraction slit pattern due to a narrow slit of width e differs much from the interference pattern due to a pair of narrow slits of separation e. In the diffraction pattern, the principal maximum is the brightest, and has non-symmetrical weak subsidiary maxima on either side. It has an angular halfwidth of λ/e. Thus narrower the slit, broader is the maximum.

In the interference pattern, the maxima & minima are equidistant and equally wide and all the maxima have the same Intensity. The width of the fringes increases, as the separation between the slit decreases.