Linear Difference Equations

Definition — Let there be a functional relationship between the independent variable x, dependent variable $f(x)$ and a number of constants $k_1, k_2 \ldots k_n$ of the form

$$P[f(x), x, k_1, k_2, \ldots, k_n] = 0 \qquad (1)$$

On applying certain number of operations on (1) we can eliminate the constants $k_1, k_2, \ldots, k_n$ from the equation (1) and get an equation containing x, $f(x)$ and various order differences of $f(x)$. For example suppose we have one constant k_1 i.e.

$$P[f(x), x, k_1] = 0 \qquad (2)$$

Operating Δ on (2), we get

$$\Delta P[f(x), x, k_1] = 0 \qquad (3)$$

Eliminating k_1 from (2) and (3) we get a relation of the form

$$F[f(x), x, \Delta f(x)] = 0 \qquad (4)$$

In general an equation of n-constants $k_1, k_2, \ldots, k_n$ can be expressed as

$$F[f(x), x, \Delta f(x), \ldots, \Delta^n f(x)] = 0 \qquad (5)$$

Further expressing $\Delta^n f(x), \ldots, \Delta f(x)$ in terms of $f(x+n), \ldots, f(x+1)$ we have

$$F[x, f(x), f(x+1), \ldots, f(x+n)] = 0 \qquad (6)$$

The relations given by (5) and (6) are known as difference equation.

Thus we conclude that a difference equation is a functional relation between the independent variable x, dependent variable $f(x)$ and the successive differences of the dependent variable.

2. Properties of Difference Equations –

The order of a difference equation is defined to be the difference between the highest and the lowest subscripts (arguments) of $f(x)$ or the power of the highest order difference in the equation. For example the order of the difference equation defined by (5) or (6) is n.

The degree of a difference equation is defined to be the highest power of $f(x)$. For example in the equation

$$u_{x+1}^4 - 2u_{x+2}^2 u_x + 4u_{x+3} = K(x) \qquad (1)$$

the highest power is 4. Hence the degree of D.E.(1) is 4 and its order is $(x+3-x)=3$.

A difference equation (D.E) is said to be Linear if its degree is one. The most general form of a linear D.E is

$$f(x+n) + p_1(x) f(x+n-1) + \dots + p_n(x) f(x) = K(x) \qquad (2)$$

where $p_1(x), p_2(x), \dots, p_n(x)$ can be some absolute constants or periodic functions of x. $K(x)$ is some function of independent variable x. The equation (2) is said to be Linear Homogeneous Difference Equation of order n if $K(x)=0$, otherwise it is known as Linear Non-Homogeneous Difference Equation.

3. Solution of a Difference Equation —

The process of finding a non-identical zero function $f(x)$ with the help of a given D.E. is known as the solution of the D.E. The solution of a D.E. is the value of dependent variable as a function of independent variable that satisfy the given equation. For example,

$$f(x) = c_1 + c_2 6^x$$

is a general solution of $f(x+2) - 7f(x+1) + 6f(x) = 0$

The solution $f(x) = c_1 + c_2 6^x$ will become the particular solution of the given D.E. if specific values of c_1 and c_2 are known, otherwise for arbitrary values of c_1 and c_2 it is known as general solution.

The D.E. $f(x+2) - 7f(x+1) + 6f(x) = 0$ can also be written as

$$(E^2 - 7E + 6)f(x) = 0$$

The equation $E^2 - 7E + 6 = 0$ is known as auxiliary equation of the given difference Equation.

4. Linear Homogeneous D.E. with constant coefficient —

A linear homogeneous difference equation with constant coefficients is given by

$$a_0 f(x) + a_1 f(x+1) + \dots + a_n f(x+n) = 0$$

or $\quad (a_0 + a_1 E + a_2 E^2 + \dots + a_n E^n) f(x) = 0 \qquad (1)$

where $a_0, a_1, a_2, \dots, a_n$ are certain constants. The methods for the solution of different forms of D.E (1) are as follows —

Case-1 Auxiliary Equation has n distinct real roots. i.e. the D.E. is of the form

$$(E-k_1)(E-k_2)\dots(E-k_n)f(x)=0 \qquad (2)$$

then the solution of the given equation is

$$\boxed{f(x)=C_1 k_1^x + C_2 k_2^x + \dots + C_n k_n^x} \qquad (3)$$

Case-2 Auxiliary equation has repeated real roots i.e. the D.E. is of the form

$$(E-a)^n f(x)=0 \qquad (4)$$

then the solution of the given equation is

$$\boxed{f(x)= a^x (C_0 + C_1 x + C_2 x^2 + \dots + C_{n-1} x^{n-1})} \qquad (5)$$

Case-3 Auxiliary equation has non-repeated complex roots say

$$r_1 = r(\cos\theta + i\sin\theta)$$
$$r_2 = r(\cos\theta - i\sin\theta)$$

then the solution of the given D.E. will be

$$\boxed{f(x)= \lambda r^x [\cos(\theta x + D)]} \qquad (6)$$

Exercise-1 Solve the difference equation

$$2y_{k+2} - 5y_{k+1} + 2y_k = 0$$

Also find the particular solution when $y_0=0, y_1=1$.

Solution- The give D.E can be written as

$$(2E^2-5E+2)y_k=0$$

Here the auxiliary equation is

$$2E^2-5E+2=0 \Rightarrow E=2 \;\&\; 1/2$$

The roots are real and distinct. Hence the general solution is given by

$$y_k = C_1 2^k + C_2 (1/2)^k$$

Now for $k=0$

$$y_0 = C_1 + C_2 = 0 \text{ (given)} \qquad \text{(i)}$$

for $k=1$

$$y_1 = 2C_1 + \frac{1}{2} C_2 = 1 \text{ (given)} \qquad \text{(ii)}$$

From equations (i) & (ii) we get $C_1 = 2/3$ & $C_2 = -2/3$

Thus the particular solution of given D.E is

$$y_k = \frac{2}{3} \cdot 2^k - \frac{2}{3}\left(\frac{1}{2}\right)^k$$

$$= \frac{2^{k+1}}{3}\left[1 - 2^{-2k}\right]$$

Exercise-2 Solve the D.E.

$$y_{k+4} - 4y_{k+3} + 6y_{k+2} - 4y_{k+1} + y_k = 0$$

Solution — The auxiliary equation of the given D.E. is

$$E^4 - 4E^3 + 6E^2 - 4E + 1 = 0$$

$$\Rightarrow (E-1)^4 = 0$$

Thus the given equation has repeated real roots each equal to one. Hence the general solution is given by

$$y_k = (C_1 + C_2 k + C_3 k^2 + C_4 k^3) 1^k$$

Exercise-3 Solve the D.E. $y_{k+2} + y_k = 0$ satisfying $y_0 = 0$ and $y_1 = 1$

Solution — The auxiliary equation is

$$E^2 + 1 \Rightarrow E = \pm i = \cos\frac{\pi}{2} \pm i \sin\frac{\pi}{2}$$

$$\left[\begin{array}{l} \text{As } 0+i = r(\cos\theta + i\sin\theta) \ \& \ 0-i = r(\cos\theta - i\sin\theta) \\ \left.\begin{array}{l} \Rightarrow r\cos\theta = 0 \\ \& \ r\sin\theta = 1 \end{array}\right\} \Rightarrow r^2 = 1 \Rightarrow r = 1 \\ \qquad \text{So, } \cos\theta = 0 = \cos\frac{\pi}{2} \Rightarrow \theta = \pi/2 \end{array}\right.$$

So, the general solution of the given D.E is

$$y_k = \lambda \cos\left[k\frac{\pi}{2} + D\right]$$

Now for $k=0$, $y_0 = \lambda \cos D = 0$ (given) $\qquad$ (i)

and for $k=1$, $y_1 = \lambda \cos\left[\frac{\pi}{2} + D\right] = 1$ (given) $\qquad$ (ii)

From (i) and (ii) we get

$$D = \pi/2 \text{ and } \lambda = -1$$

Thus the particular solution of the equation is

$$y_k = -\cos\frac{\pi}{2}(k+1) = \sin\frac{k\pi}{2}$$

Exercise-4 Solve the D.E.

$$y_{k+2} - 2y_{k+1} + 2y_k = 0$$

Solution— The auxiliary equation is given by

$$E^2 - 2E + 2 = 0 \Rightarrow E = \frac{2 \pm \sqrt{(4-8)}}{2}$$

$$\Rightarrow E = 1 \pm i$$

To change these roots into polar form we put

$$1+i = r(\cos\theta + i\sin\theta)$$
$$1-i = r(\cos\theta - i\sin\theta)$$

on equating real & imaginary parts we get

$$r\cos\theta = 1 \quad \text{(i)}$$

and $r\sin\theta = 1$ (ii)

on squaring & adding (i) & (ii), we get

$$r^2 = 2 \Rightarrow r = \sqrt{2}$$

$$\therefore \cos\theta = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$$

Now the solution of the given D.E is

$$y_k = \lambda(\sqrt{2})^k \left[\cos\left(k.\frac{\pi}{4} + D\right)\right]$$

Exercise-5 Solve the following D.E's —

(i) $y_{k+2} - 5y_{k+1} + 6y_k = 0$

(ii) $u_{x+2} - 7u_{x+1} + 12u_x = 0$

(iii) $u_{n+2} + 8u_{n+1} + 16u_n = 0$

(iv) $3y_{x+2} - 6y_{x+1} + 4y_x = 0$

(v) $y_{k+2} + 6y_{k+1} + 25y_k = 0$

Also find the particular solution in each case when the initial condition is $y_0 = u_0 = 0$ & $y_1 = u_1 = 1$.

(Try your self)

Exercise-6 Reduce the following difference equations to linear forms. Also find their orders and solve them.

(i) $\Delta^2 y_k + 3\Delta y_k - 3y_k = 0$

(ii) $\Delta^3 y_k + \Delta^2 y_k + \Delta y_k + y_k = 0$

Solutions – We have

$$\Delta y_k = y_{k+1} - y_k$$

$$\Delta^2 y_k = \Delta[\Delta y_k] = \Delta[y_{k+1} - y_k]$$

$$= (y_{k+2} - y_{k+1}) - (y_{k+1} - y_k) = y_{k+2} - 2y_{k+1} + y_k$$

and $\Delta^3 y_k = \Delta[\Delta^2 y_k]$

$$= \Delta[y_{k+2} - 2y_{k+1} + y_k]$$

$$= (y_{k+3} - 2y_{k+2} + y_{k+1}) - (y_{k+2} - 2y_{k+1} + y_k)$$

$$= y_{k+3} - 3y_{k+2} + 3y_{k+1} - y_k$$

(i) Substituting the values of Δy_k, $\Delta^2 y_k$ in first D.E. we get

$$y_{k+2} - 2y_{k+1} + y_k + 3(y_{k+1} - y_k) - 3y_k = 0$$

or $y_{k+2} + y_{k+1} - 5y_k = 0$

The order of this D.E is 2.

Now the D.E can be solved.

(ii) Substituting the values of Δy_k, $\Delta^2 y_k$ and $\Delta^3 y_k$ in second D.E. we have

$$(y_{k+3} - 3y_{k+2} + 3y_{k+1} - y_k) + (y_{k+2} - 2y_{k+1} + y_k) + (y_{k+1} - y_k) + y_k = 0$$

$$\Rightarrow y_{k+3} - 2y_{k+2} + 2y_{k+1} = 0$$

The order of this D.E is also 2.

Now it can be solved easily.

5 Linear Homogeneous D.E.s. with Variable Constants –

Let the D.E. be of the form

$$f(x+1) - P(x)f(x) = 0 \qquad (1)$$

where $P(x)$ is some function of x.

Putting $x = 0, 1, 2, \ldots, x-1$ in (1) successively, we get

$$f(1) = P(0)f(0)$$

$$f(2) = P(1)f(1)$$
$$= P(1)P(0)f(0)$$

$$f(3) = P(2)f(2)$$
$$= P(2)P(1)P(0)f(0)$$

In general

$$f(x) = P(x-1)P(x-2)\cdots P(1)\cdot P(0)\cdot f(0)$$
$$= f(0)\prod_{r=0}^{x-1} P(r) = C\prod_{r=0}^{x-1} P(r) \qquad (2)$$

where C is the value of function at $x=0$.

Exercise-1 Solve the equation

$$u_{x+1} = 2^x u_x$$

Solution– The D.E. is of the form (1) where $P(x) = 2^x$

Hence using (2), the solution will be

$$u_x = C\prod_{r=0}^{x-1} 2^r = C\cdot 2^{0+1+2+\cdots+x-1}$$
$$= C\cdot 2^{x(x-1)/2}$$

Exercise-2 Solve the following difference equations

(a) $u_{x+1} - \frac{1}{x}u_x = 0$; $x > 0$

(b) $u_{x+1} - 3^x u_x = 0$

(c) $u_{x+1} - e^x u_x = 0$

(d) $u_{x+1} - a^x u_x = 0$

6. Non Homogeneous L D Es. with Constant Coefficients —

The general form of non-homogeneous linear D.E with constant coefficients is given by

$$a_0 f(x) + a_1 f(x+1) + \dots + a_n f(x+n) = Q(x)$$

or $(a_0 + a_1 E + a_2 E^2 + \dots + a_n E^n) f(x) = Q(x)$ (1)

or $\boxed{f(x) = \frac{Q(x)}{K(E)}}$ (2)

where $K(E) = a_0 + a_1 E + \dots + a_n E^n$

is known as auxiliary function.

Expression (2) provides us a particular solution of D.E. (1). The general solution of the given non-homogeneous D.E is found by adding a particular solution of the given non-homogeneous linear D.E and the general solution of the associated homogeneous equation.

That is

General solution of non-homogeneous DE
= General solution of corresponding homogeneous DE + Particular solution of non-homogeneous D.E.

The general solution of the associated homogeneous D.E. i.e. of $K(E) f(x) = 0$ can be obtained by any of the method discussed in section-4. Now we develop the methods for finding a particular solutions of the non-homogeneous D.E of the form (1) for different forms of the function $Q(x)$.

Case-I Let $Q(x)$ = Some constant say A, then the particular solution of D.E.(1) will be

$$\boxed{f(x) = \frac{A}{K(1)}}, \quad K(1) \neq 0 \qquad (3)$$

$K(1)$ is obtained by putting $K=1$ in $K(E)$.

Case-II Let $Q(x)$ be of the form AB^x, where A & B are certain constant. In this case the particular solution of (1) will be

$$\boxed{f(x) = \frac{AB^x}{K(B)}} \quad \text{provided } K(B) \neq 0 \qquad (4)$$

Case-III Let $Q(x)$ be a polynomial of degree m in x. In this case the particular solution of (1) will be

$$f(x) = \frac{Q(x)}{K(1+\Delta)} \qquad (5)$$

$$\Rightarrow \boxed{f(x) = R(\Delta) \cdot Q(x)}$$

where $R(\Delta)$ = function of $\Delta = [K(1+\Delta)]^{-1}$

Case-IV Let $Q(x)$ be of the form $B^x V(x)$, where B is some constant and $V(x)$ is a polynomial of degree m in x. In this case the particular solution of (1) will be

$$f(x) = \frac{B^x V(x)}{K(B+B\Delta)}$$

$$\Rightarrow \boxed{f(x) = B^x \{K(B+B\Delta)\}^{-1} V(x)} \qquad (6)$$

Theorem- If $g(x)$ is a solution of the nth order homogeneous D.E. with constant coefficients and $G(x)$ is the solution of the corresponding non-homogeneous D.E, then $g(x) + G(x)$ is also a solution of the given non-homogeneous D.E.

Proof- Let the non-homogeneous D.E. be

$$a_0 f(x) + a_1 f(x+1) + \cdots + a_n f(x+n) = Q(x) \qquad (1)$$

and the corresponding homogeneous D.E is

$$a_0 f(x) + a_1 f(x+1) + \cdots + a_n f(x+n) = 0 \qquad (2)$$