Non-Central Chi-Square

Definition – If X_i $(i=1,2,\dots n)$ are n mutually independent variates having normal distribution with mean μ_i and common variance 'unity' $\forall\ i=1,2,\dots n$ then

$$U=\sum_{i=1}^{n} X_i^2$$

is distributed as a non-central chi-square with n d.f. and non-centrality parameter $\lambda^2=\sum_{i=1}^{n}\mu_i^2$. It is denoted by

$$U \sim \chi'^2_n(\lambda^2)$$

Derivation – (Ist Method)

Let $\underset{\sim}{Q}$ be a $n \times n$ orthogonal matrix with elements of the first row being $\left(\frac{\mu_1}{\lambda}, \frac{\mu_2}{\lambda}, \dots, \frac{\mu_n}{\lambda}\right)$ and let $\underset{\sim}{Z}=\underset{\sim}{Q}\underset{\sim}{X}$ be an orthogonal transformation then

$$\begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} q_{11}=\frac{\mu_1}{\lambda} & q_{12}=\frac{\mu_2}{\lambda} & \dots & q_{1n}=\frac{\mu_n}{\lambda} \\ q_{21} & q_{22} & \dots & q_{2n} \\ \dots & \dots & \dots & \dots \\ q_{n1} & q_{n2} & \dots & q_{nn} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$$

$$\Rightarrow Z_i = \sum_{j=1}^{n} q_{ij} X_j \quad ; i=1,2,\dots n$$

$$\therefore E(Z_i) = \sum_{j=1}^{n} q_{ij}E(X_j) = \sum_{j=1}^{n} q_{ij}\frac{q_{1j}}{q_{1j}}E(X_j)$$

$$= \sum_{j=1}^{n} q_{ij}\frac{q_{1j}}{\mu_j/\lambda}\cdot\mu_j = \lambda\sum_{j=1}^{n} q_{ij}q_{1j} = \lambda\delta_{i1}$$

$$= \begin{cases} \lambda & \text{if } i=1 \\ 0 & \text{otherwise} \end{cases}$$

$$V(Z_i) = \sum_{j=1}^{n} q_{ij}^2 V(X_j) = \sum_{j=1}^{n} q_{ij}^2 = 1 \quad \forall\ i=1,2,\dots,n$$

Thus $\underset{\sim}{Z} \sim N(\underset{\sim}{\nu}, \underset{\sim}{I})$ where $\underset{\sim}{\nu}=\begin{pmatrix}\lambda \\ 0 \\ \vdots \\ 0\end{pmatrix}$

Now by the property of orthogonal transformation

$$U=\sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} Z_i^2 = Z_1^2+\sum_{i=2}^{n} Z_i^2 = Z_1^2 + W \text{ (say)}$$

Thus $Z_1 \sim N(\lambda, 1)$ and $W=\sum_{i=2}^{n} Z_i^2 \sim \chi^2_{n-1}$ (central)

The joint distribution of Z_1 and W is

$$dF(z_1, w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_1-\lambda)^2} \frac{e^{-\frac{w}{2}} w^{\frac{n-1}{2}-1}}{2^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} dz_1\, dw \qquad \begin{matrix} -\infty < z_1 < \infty \\ 0 \leq w < \infty \end{matrix}$$

$$= c\, e^{-\frac{1}{2}(\lambda^2 + z_1^2 + w)} w^{\frac{n-3}{2}} e^{\lambda z_1} dz_1\, dw \quad , \quad c = \frac{1}{2^{n/2}\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}$$

$$= c\, e^{-\frac{1}{2}(\lambda^2 + z_1^2 + w)} w^{\frac{n-3}{2}} \sum_{\alpha=0}^{\infty} \frac{(\lambda z_1)^\alpha}{\alpha!} dz_1\, dw$$

The joint distribution of $U = W + Z_1^2$ and Z_1 can be obtained by substituting $w = u - z_1^2 \Rightarrow dw = du$

$$\therefore\ dF(z_1, u) = c\, e^{-\frac{1}{2}(\lambda^2+u)} (u - z_1^2)^{\frac{n-3}{2}} \sum_{\alpha=0}^{\infty} \frac{\lambda^\alpha z_1^\alpha}{\alpha!} dz_1\, du \qquad (1)$$

When we integrate (1) w.r.t. z_1 term by term, The integrals will be zere for all the odd values of α. Since for odd α, the integrands will be odd function of z_1. Thus for even α, the distribution of U is

$$dF(u) = c\, e^{-\frac{1}{2}(\lambda^2+u)} \sum_{\beta=0}^{\infty} \frac{\lambda^{2\beta}}{(2\beta)!} \int_{-\infty}^{\infty} z_1^{2\beta} (u - z_1^2)^{\frac{n-3}{2}} dz_1$$

Putting $z_1 = t\sqrt{u} \Rightarrow dz_1 = \sqrt{u}\, dt$

For limits of t, we have

$w \geq 0 \Rightarrow u - z_1^2 \geq 0 \Rightarrow u - t^2 u \geq 0$

$\Rightarrow u(1-t^2) \geq 0 \Rightarrow u \geq 0$ and $1 - t^2 \geq 0$

$\Rightarrow u \geq 0$ and $t^2 \leq 1 \Rightarrow u \geq 0$ and $-1 < t < +1$

$$\therefore\ dF(u) = c\, e^{-\frac{1}{2}(\lambda^2+u)} du \sum_{\beta=0}^{\infty} \frac{\lambda^{2\beta}}{(2\beta)!} \int_{-1}^{1} (t\sqrt{u})^{2\beta} (u - ut^2)^{\frac{n-3}{2}} \sqrt{u}\, dt$$

$$= c\, e^{-\frac{1}{2}(\lambda^2+u)} u^{\frac{n}{2}-1} du \sum_{\beta=0}^{\infty} \frac{\lambda^{2\beta}}{(2\beta)!} u^\beta \int_{-1}^{1} t^{2\beta} (1-t^2)^{\frac{n-3}{2}} dt$$

Consider $I = \int_{-1}^{1} t^{2\beta} (1-t^2)^{\frac{n-3}{2}} dt = 2\int_0^1 t^{2\beta} (1-t^2)^{\frac{n-3}{2}} dt$

Putting $t^2 = y \Rightarrow dt = \frac{dy}{2\sqrt{y}}$

$$\therefore\ I = 2\int_0^1 y^{\beta - \frac{1}{2}} (1-y)^{\frac{n-1}{2}-1} \frac{dy}{2} = B[(\beta + \tfrac{1}{2}), \tfrac{n-1}{2}]$$

Now

$$dF(u) = c\, e^{-\frac{1}{2}(\lambda^2+u)} u^{\frac{n}{2}-1} \sum_{\beta=0}^{\infty} \frac{\lambda^{2\beta}}{(2\beta)!} u^\beta B[(\beta+\tfrac{1}{2}), \tfrac{n-1}{2}]\, du$$

$$= \frac{1}{2^{\frac{n}{2}}\sqrt{\pi}\,\cancel{\Gamma(\frac{n-1}{2})}} e^{-\frac{1}{2}(\lambda^2+u)} u^{\frac{n}{2}-1} \sum_{\beta=0}^{\infty} \frac{\lambda^{2\beta}}{(2\beta)!} u^\beta \frac{\Gamma(\beta+\frac{1}{2})\cancel{\Gamma(\frac{n-1}{2})}}{\Gamma(\beta+\frac{n}{2})} du$$

$$= \frac{e^{-\frac{1}{2}(\lambda^2+u)} u^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\sqrt{\pi}} \sum_{\beta=0}^{\infty} \frac{(\lambda^2)^\beta u^\beta \sqrt{\pi}\, \cancel{\Gamma(2\beta)}}{2\beta \cancel{\Gamma(2\beta)}\, \Gamma(\beta+\frac{n}{2})\, 2^{2\beta-1}\Gamma(\beta)}$$

$$\because \Gamma(\beta+\tfrac{1}{2}) = \frac{\sqrt{\pi}\,\Gamma(2\beta)}{2^{2\beta-1}\Gamma(\beta)}$$

$$= \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2}}(\frac{\lambda^2}{2})^\beta}{\beta!} \cdot \frac{e^{-\frac{u}{2}} u^{\frac{n}{2}+\beta-1}}{2^{\frac{n}{2}+\beta}\Gamma(\beta+\frac{n}{2})} \quad du$$

$$= \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2}}(\frac{\lambda^2}{2})^\beta}{\beta!} f(\chi^2_{n+2\beta}).$$

This is the density of the non-central χ^2 distribution with n degrees of freedom and non-centrality parameter λ^2.

<u>IInd Method</u> (By M.G.F.)

If x_i $(i=1,2,\ldots,n)$ are independently dis^d $\sim N(\mu_i, 1)$ then the m.g.f. of non-central χ^2 variate $U=\sum_{i=1}^n x_i^2$ is given by

$$M_U(t) = M_{\sum_{i=1}^n x_i^2}(t) = \prod_{i=1}^n M_{x_i^2}(t) \qquad (1)$$

Now

$$M_{x_i^2}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t x_i^2} e^{-(x_i-\mu_i)^2/2}\, dx_i$$

Consider $e^{t x_i^2 - (x_i-\mu_i)^2/2} = \exp[-\{(\frac{1}{2}-t)x_i^2 - \mu_i x_i + \frac{\mu_i^2}{2}\}]$

$$= \exp[-(\tfrac{1-2t}{2})\{x_i^2 - \frac{2\mu_i x_i}{1-2t} + \frac{\mu_i^2}{1-2t}\}]$$

$$= \exp[-(\tfrac{1-2t}{2})\{(x_i - \frac{\mu_i}{1-2t})^2 + \frac{\mu_i^2}{1-2t} - \frac{\mu_i^2}{(1-2t)^2}\}]$$

$$= \exp[-(\tfrac{1-2t}{2})\{(x_i - \frac{\mu_i}{1-2t})^2 + \frac{\mu_i^2(\cancel{1}-2t-\cancel{1})}{(1-2t)^2}\}]$$

$$= \exp(\frac{t\mu_i^2}{1-2t}) \exp[-(\tfrac{1-2t}{2})(x_i - \frac{\mu_i}{1-2t})^2]$$

$$\therefore M_{x_i^2}(t) = \exp(\frac{t\mu_i^2}{1-2t}) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp[-(\tfrac{1-2t}{2})(x_i - \frac{\mu_i}{1-2t})^2]\, dx_i$$

Putting $\sqrt{(1-2t)}(x_i - \frac{\mu_i}{1-2t}) = y_i \Rightarrow dx_i = \frac{dy_i}{(1-2t)^{1/2}}$

Therefore.

$$M_{x_i^2}(t) = \exp(\frac{t\mu_i^2}{1-2t}) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y_i^2}{2}} \frac{dy_i}{(1-2t)^{1/2}}$$

$$= (1-2t)^{-1/2} \exp(\frac{t\mu_i^2}{1-2t})$$

Now equation (1) becomes

$$M_U(t) = \prod_{i=1}^n (1-2t)^{-1/2} \exp(\frac{t\mu_i^2}{1-2t}) = (1-2t)^{-\frac{n}{2}} \exp[\frac{t}{1-2t}\sum_{i=1}^n \mu_i^2]$$

$$= (1-2t)^{-\frac{n}{2}} \exp[\frac{t\lambda^2}{1-2t}] = (1-2t)^{-\frac{n}{2}} \exp[\frac{\lambda^2}{2}(\frac{2t+1-1}{1-2t})]$$

$$= (1-2t)^{-\frac{n}{2}} e^{-\lambda^2/2} \cdot \exp\left(\frac{\lambda^2/2}{1-2t}\right) \cdot$$

$$= (1-2t)^{-\frac{n}{2}} e^{-\lambda^2/2} \sum_{\beta=0}^{\infty} \left(\frac{\lambda^2/2}{1-2t}\right)^{\beta} \frac{1}{\beta!}$$

$$= \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2}} \left(\frac{\lambda^2}{2}\right)^{\beta}}{\beta!} (1-2t)^{-\left(\frac{n}{2}+\beta\right)}$$

Hence by uniqueness theorem of m.g.f.'s the pdf of non-central chi-square variate with n d.f. and non-centrality parameter λ^2 is given by

$$f(u) = \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2}} \left(\frac{\lambda^2}{2}\right)^{\beta}}{\beta!} \text{ pdf of central } \chi^2_{n+2\beta}$$

$$= \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2}} \left(\frac{\lambda^2}{2}\right)^{\beta}}{\beta!} \frac{e^{-\frac{u}{2}} u^{\frac{n+2\beta}{2}-1}}{2^{\frac{n+2\beta}{2}} \Gamma\left(\frac{n+2\beta}{2}\right)}, \quad 0 \leq u < \infty$$

Moments— The rth raw moment is given by

$$\mu'_r = E(U)^r = \int_0^{\infty} u^r f(u)\, du$$

$$= \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2}} \left(\frac{\lambda^2}{2}\right)^{\beta}}{\beta!} \int_0^{\infty} \frac{e^{-\frac{u}{2}} u^{\frac{n+2\beta}{2}+r-1}}{2^{\frac{n+2\beta}{2}} \Gamma\left(\frac{n+2\beta}{2}\right)} du.$$

$$= \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2}} \left(\frac{\lambda^2}{2}\right)^{\beta}}{\beta!} \frac{1}{\cancel{2^{\frac{n+2\beta}{2}}} \Gamma\left(\frac{n+2\beta}{2}\right)} \frac{\Gamma\left(\frac{n+2\beta}{2}+r\right)}{\left(\frac{1}{2}\right)^{\cancel{\frac{n+2\beta}{2}}+r}}$$

$$= \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2}} \left(\frac{\lambda^2}{2}\right)^{\beta}}{\beta!} 2^r \left(\frac{n+2\beta}{2}\right)^{[r]}$$

$$\mu'_1 = \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2}} \left(\frac{\lambda^2}{2}\right)^{\beta}}{\beta!} \cancel{2} \cdot \left(\frac{n+2\beta}{\cancel{2}}\right) = n + 2\sum_{\beta=1}^{\infty} \frac{e^{-\frac{\lambda^2}{2}} \left(\frac{\lambda^2}{2}\right)^{\beta}}{\cancel{\beta}(\beta-1)!} \cancel{\beta}$$

$$= n + 2 \cdot \frac{\lambda^2}{2} \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2}} \left(\frac{\lambda^2}{2}\right)^{j}}{j!} = n + \lambda^2 = E(U)$$

$$\mu'_2 = \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2}} \left(\frac{\lambda^2}{2}\right)^{\beta}}{\beta!} \cancel{2^2} \cdot \left(\frac{n+2\beta}{\cancel{2}}\right)\left(\frac{n+2\beta+2}{\cancel{2}}\right)$$

$$= \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2}} \left(\frac{\lambda^2}{2}\right)^{\beta}}{\beta!} \left(n^2 + 2n + 4\beta^2 + 4\beta(n+1)\right)$$

$$= (n^2+2n) + 4(n+1)\sum_{\beta=0}^{\infty} \beta \frac{e^{-\frac{\lambda^2}{2}}\left(\frac{\lambda^2}{2}\right)^\beta}{\beta!} + 4\sum_{\beta=0}^{\infty} \beta^2 \frac{e^{-\frac{\lambda^2}{2}}\left(\frac{\lambda^2}{2}\right)^\beta}{\beta!}$$

$$= (n^2+2n) + 4(n+1)\frac{\lambda^2}{2} + 4\frac{\lambda^2}{2}\left(\frac{\lambda^2}{2}+1\right)$$

$\because$ For Poisson dis^n $\mu_1' = \lambda$, $\mu_2' = \lambda(\lambda+1)$

$$= n^2 + 2n + 2(n+1)\lambda^2 + \lambda^4 + 2\lambda^2$$

$$= \lambda^4 + 2(n+2)\lambda^2 + n(n+2)$$

Therefore

$$V(U) = \mu_2' - \mu_1'^2 = \lambda^4 + 2(n+2)\lambda^2 + n(n+2) - n^2 - \lambda^4 - 2n\lambda^2$$

$$= 4\lambda^2 + 2n$$

As $\lambda^2 = 0$, $E(U) = n$ and $V(U) = 2n$

Remark— In the probability density f^n of non-central chi-square variate, if we put

$$\frac{u}{2} = x \Rightarrow du = 2dx$$

then

$$dF(x) = \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2}}\left(\frac{\lambda^2}{2}\right)^\beta}{\beta!} \frac{e^{-x}(\cancel{2}x)^{\frac{n+2\beta}{2}-1}}{\cancel{2^{\frac{n+2\beta}{2}}}\,\Gamma\left(\frac{n+2\beta}{2}\right)} \cancel{2}\,dx$$

$$= \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2}}\left(\frac{\lambda^2}{2}\right)^\beta}{\beta!} \frac{e^{-x} x^{\frac{n+2\beta}{2}-1}}{\Gamma\left(\frac{n+2\beta}{2}\right)} dx$$

which is the p.d.f. of non-central gamma variate with parameter $\left(\frac{n}{2}+\beta\right)$.

Additive or Re-productive property—

Statement— If U_i $(i=1,2,\cdots,k)$ are independent non-central chi-square variates with n_i d.f. and non-centrality element λ_i^2, then $\sum_{i=1}^{k} U_i$ is also a non central chi-square variate with $\sum_{i=1}^{k} n_i$ d.f and non-centrality parameter $\lambda = \sum_{i=1}^{k}\lambda_i^2$

Proof— We have

$$M_{U_i}(t) = (1-2t)^{-\frac{n_i}{2}} e^{\frac{t\lambda_i^2}{1-2t}} \qquad i = 1,2,\cdots k$$

Therefore

$$M_{\sum_{i=1}^{k} U_i}(t) = \prod_{i=1}^{k} M_{U_i}(t)$$

$$= (1-2t)^{-\frac{1}{2}\sum_{i=1}^{k} n_i}\, e^{\frac{t}{1-2t}\sum_{i=1}^{k}\lambda_i^2}$$

e.

which is the m.g.f. of a non-central chi-square variate with $\sum_{i=1}^{k} n_i$ d.f. and non-centrality parameter $\lambda^2 = \sum_{i=1}^{k} \lambda_i^2$. Hence by uniqueness theorem

$$\sum_{i=1}^{k} U_i \sim \chi'^2_{\sum_{i=1}^{k} n_i} \left(\sum_{i=1}^{k} \lambda_i^2 \right)$$

Proved

[14] Non Central F-Distribution

Definition – If V has a non-central chi-square distribution with n d.f and non-centrality parameter λ and W has an independent central chi-square with m d.f. then the ratio

$$U = \frac{V/n}{W/m}$$

has non-central F-distribution with (n,m) d.f. and non-centrality parameter λ.

Derivation of the distribution –

Since V and W are independent, their joint pdf is given by

$$dF(v,w) = \sum_{\beta=0}^{\infty} \frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^{\beta}}{\beta!} \frac{e^{-\frac{v}{2}}(v)^{\frac{n}{2}+\beta-1}}{2^{\frac{n+2\beta}{2}}\Gamma\left(\frac{n+2\beta}{2}\right)} \times \frac{e^{-\frac{w}{2}}(w)^{\frac{m}{2}-1}}{2^{\frac{m}{2}}\Gamma\left(\frac{m}{2}\right)}\, dv\, dw \qquad \begin{matrix} 0 \le v < \infty \\ 0 \le w < \infty \end{matrix}$$

Putting $v = \frac{nuw}{m} \Rightarrow dv = \frac{nw}{m}\,du$

$$\therefore\ dF(u,w) = \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^{\beta} e^{-\frac{nuw}{2m}} \left(\frac{nuw}{m}\right)^{\frac{n}{2}+\beta-1}}{\beta!\ 2^{\frac{n+m}{2}+\beta}\,\Gamma\left(\frac{n+2\beta}{2}\right)\Gamma\left(\frac{m}{2}\right)}\, e^{-\frac{w}{2}}(w)^{\frac{m}{2}-1} \cdot \frac{nw}{m}\, du\, dw$$

$$= \frac{n}{m}\sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^{\beta}\left(\frac{nu}{m}\right)^{\frac{n}{2}+\beta-1} w^{\frac{n+m}{2}+\beta-1}}{\beta!\ 2^{\frac{n+m}{2}+\beta}\,\Gamma\left(\frac{n+2\beta}{2}\right)\Gamma\left(\frac{m}{2}\right)}\, e^{-\frac{w}{2}\left(1+\frac{nu}{m}\right)}\, du\, dw \qquad (1)$$

Integrating (1) w.r.t. w over its range, we have

$$dF(u) = \frac{n}{m}\sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^{\beta}\left(\frac{nu}{m}\right)^{\frac{n}{2}+\beta-1} du}{\beta!\ 2^{\frac{n+m}{2}+\beta}\,\Gamma\left(\frac{n+2\beta}{2}\right)\Gamma\left(\frac{m}{2}\right)} \int_0^{\infty} e^{-\frac{w}{2}\left(1+\frac{nu}{m}\right)} w^{\frac{n+m}{2}+\beta-1}\, dw$$

$$= \frac{n}{m}\sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^{\beta}\left(\frac{nu}{m}\right)^{\frac{n}{2}+\beta-1} du}{\beta!\ 2^{\frac{n+m}{2}+\beta}\,\Gamma\left(\frac{n+2\beta}{2}\right)\Gamma\left(\frac{m}{2}\right)} \frac{\Gamma\left(\frac{n+m}{2}+\beta\right)}{\left[\frac{1}{2}\left(1+\frac{nu}{m}\right)\right]^{\frac{n+m}{2}+\beta}}$$

$$= \frac{n}{m}\sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^{\beta}\left(\frac{nu}{m}\right)^{\frac{n}{2}+\beta-1}}{\beta!\ B\left(\frac{n}{2}+\beta, \frac{m}{2}\right)} \cdot \frac{1}{\left(1+\frac{nu}{m}\right)^{\frac{n+m}{2}+\beta}}\, du$$

$$= \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^{\beta}}{\beta!} f(u)\, du \qquad \text{where } U \sim F\left(\frac{n+2\beta}{}, \frac{m}{}\right)$$

Remark — For $\lambda = 0$, we get

$$dF(u) = \frac{n}{m}\frac{\left(\frac{n}{m}u\right)^{\frac{n}{2}-1}\, du}{B\left(\frac{n}{2},\frac{m}{2}\right)\left(1+\frac{n}{m}u\right)^{\frac{n+m}{2}}}, \qquad 0 \le u < \infty$$

Moments of non-central F-distribution -

Suppose U follows non-central F-distribution with (n, m) d.f. and non-centrality parameter λ, then the rth raw moment of the statistic U is

$$E(U^r) = E\left[\frac{m}{n}\frac{V}{W}\right]^r, \quad \text{where } V \sim \text{Non-Central } \chi^2_n(\lambda)$$
$$\& \quad W \sim \text{Central } \chi^2_m$$

$$= \left(\frac{m}{n}\right)^r E(V)^r . E(W)^{-r} \qquad \because V \& W \text{ are indep.}$$

Now

$$E(V)^r = \int_0^\infty v^r f(v)\, dv$$

$$= \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^\beta \Gamma\left(\frac{n+r}{2}+\beta\right)}{\beta!\, 2^{-r/2}\, \Gamma\left(\frac{n}{2}+\beta\right)}$$

and $E(W)^{-r} = \int_0^\infty w^{-r} f(w)\, dw$

$$= \int_0^\infty w^{-r} \frac{e^{-\frac{w}{2}} w^{\frac{m}{2}-1}}{2^{\frac{m}{2}}\Gamma\left(\frac{m}{2}\right)}\, dw$$

$$= \frac{1}{2^{\frac{m}{2}}\Gamma\left(\frac{m}{2}\right)} \int_0^\infty e^{-\frac{w}{2}} w^{\frac{m}{2}-r-1}\, dw$$

$$= \frac{1}{2^{\frac{m}{2}}\Gamma\left(\frac{m}{2}\right)} \frac{\Gamma\left(\frac{m}{2}-r\right)}{\left[\frac{1}{2}\right]^{\frac{m}{2}-r}} = \frac{\Gamma\left(\frac{m}{2}-r\right)}{2^r\, \Gamma\left(\frac{m}{2}\right)}$$

$$= 1/2^r \left(\frac{m}{2}-1\right)^{(r)}$$

$$\therefore \quad E(U^r) = \left(\frac{m}{n}\right)^r \frac{1}{2^r\left(\frac{m}{2}-1\right)^{(r)}} \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^\beta \Gamma\left(\frac{n+r}{2}+\beta\right)}{\beta!\, 2^{-r/2}\, \Gamma\left(\frac{n}{2}+\beta\right)}$$

[15] Non-Central Beta Distributions

Definition – If U has a non-central chi-square distribution with n_1 d.f. and non-centrality parameter λ and V (independent of U) has an central chi-square distribution with n_2 d.f. then the ratio

$$X = \frac{U}{U+V}$$

has non-central beta distribution of I kind with parameters $\left(\frac{n_1}{2}, \frac{n_2}{2}, \lambda\right)$

Derivation – Since U and V are independent, their joint distribution is given by

$$dF(u,v) = \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^{\beta}}{\beta!} \frac{e^{-\frac{u}{2}} u^{\frac{n_1}{2}+\beta-1}}{2^{\frac{n_1}{2}+\beta}\Gamma\left(\frac{n_1}{2}+\beta\right)} \frac{e^{-\frac{v}{2}} v^{\frac{n_2}{2}-1}}{2^{\frac{n_2}{2}}\Gamma\left(\frac{n_2}{2}\right)} \, du\, dv$$

Putting $x = \frac{u}{u+v} \Rightarrow x(u+v) = u$

$\Rightarrow u = \frac{x}{1-x} v \Rightarrow du = \frac{v}{(1-x)^2} dx$

$$\therefore dF(x,v) = \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^{\beta} e^{-\frac{vx}{2(1-x)}} \left(\frac{vx}{1-x}\right)^{\frac{n_1}{2}+\beta-1}}{\beta!\; 2^{\frac{n_1}{2}+\beta}\Gamma\left(\frac{n_1}{2}+\beta\right)\; 2^{\frac{n_2}{2}}\Gamma\left(\frac{n_2}{2}\right)} e^{-\frac{v}{2}} v^{\frac{n_2}{2}-1} \frac{v}{(1-x)^2} dx\, dv$$

$$\Rightarrow dF(x) = \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^{\beta}\left(\frac{x}{1-x}\right)^{\frac{n_1}{2}+\beta-1} dx}{(1-x)^2 \beta!\; 2^{\frac{n_1+n_2}{2}+\beta}\Gamma\left(\frac{n_1}{2}+\beta\right)\Gamma\left(\frac{n_2}{2}\right)} \int_0^{\infty} e^{-\frac{v}{2}\left(\frac{1}{1-x}\right)} v^{\frac{n_1+n_2}{2}+\beta-1} dv$$

$$= \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^{\beta}\left(\frac{x}{1-x}\right)^{\frac{n_1}{2}+\beta-1} dx}{(1-x)^2 \beta!\; \cancel{2^{\frac{n_1+n_2}{2}+\beta}}\Gamma\left(\frac{n_1}{2}+\beta\right)\Gamma\left(\frac{n_2}{2}\right)} \times \frac{\Gamma\left(\frac{n_1+n_2}{2}+\beta\right)}{\left[\frac{1}{\cancel{2}(1-x)}\right]^{\frac{n_1+n_2}{2}+\beta}}$$

$$= \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^{\beta}}{\beta!\; B\left(\frac{n_1}{2}+\beta, \frac{n_2}{2}\right)} x^{\frac{n_1}{2}+\beta-1}(1-x)^{\frac{n_2}{2}-1} dx \qquad (1)$$

$0 \le x \le 1$

Remark-1 For $\lambda = 0$, we get

$$dF(x) = \frac{1}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} x^{\frac{n_1}{2}-1}(1-x)^{\frac{n_2}{2}-1} dx$$

Remark-2 On putting $x = \frac{y}{y+1}$ in (1) we have

$$dF(y) = \sum_{\beta=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^{\beta}}{\beta!\; B\left(\frac{n_1}{2}+\beta, \frac{n_2}{2}\right)} \frac{y^{\frac{n_1}{2}+\beta-1}}{(1+y)^{\frac{n_1+n_2}{2}+\beta}} dy \qquad 0 \le y < \infty \qquad (2)$$

which is the distribution of non-central beta disn of II kind.

Non-Central t-Distribution

Definition — Let X be a normal variate with mean, μ and unit variance and Y^2 be an independent central chi-square variate with n degree of freedom, then the statistic

$$U = \frac{X\sqrt{n}}{Y}$$

follows a non-central t-distribution with n degrees of freedom and non-centrality parameter μ.

Derivation — In order to derive the distribution of $U = \frac{X\sqrt{n}}{Y}$, we shall first find the joint disn of X and Y and then making the transform $X = \frac{uy}{\sqrt{n}}$ in joint disn of X and Y, we can obtain the joint disn of u and Y. Now integreating the joint disn of U and Y over the range of Y, we can obtain the marginal disn of U.

We have

$$X \sim N(\mu, 1)$$

$$\therefore \; dF(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2} \quad ; \quad -\infty < x < \infty$$

and $\quad Y^2 \sim$ Central χ^2_n

$$\therefore \; dF(y^2) = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} e^{-\frac{y^2}{2}} (y^2)^{\frac{n}{2}-1} dy^2$$

$$\text{or} \quad dF(y) = \frac{1}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})} e^{-\frac{y^2}{2}} y^{n-1} dy \quad ; \quad 0 \le y < \infty$$

Therefore the joint distribution of X and Y is

$$dF(x,y) = \frac{1}{2^{\frac{n-2}{2}}\Gamma(\frac{n}{2})\sqrt{2\pi}} e^{-\frac{x^2}{2} - \frac{\mu^2}{2} + \mu x - \frac{y^2}{2}} y^{n-1} dx\, dy$$

$$= \frac{y^{n-1}}{2^{\frac{n-2}{2}}\Gamma(\frac{n}{2})\sqrt{2\pi}} e^{-\frac{x^2}{2} - \frac{y^2}{2} - \frac{\mu^2}{2}} \sum_{i=0}^{\infty} \frac{(\mu x)^i}{i!} dx\, dy \qquad (1)$$

Now making transformation $x = \frac{uy}{\sqrt{n}}$, $\Rightarrow dx = \frac{y}{\sqrt{n}} du$, in (1) we get

$$dF(u,y) = \frac{y^{n-1}}{2^{\frac{n-2}{2}}\Gamma(\frac{n}{2})\sqrt{2\pi}} e^{-\frac{u^2y^2}{2n}-\frac{y^2}{2}-\frac{\mu^2}{2}} \sum_{i=0}^{\infty} \frac{\left(\frac{\mu u y}{\sqrt{n}}\right)^i}{i!} \frac{y}{\sqrt{n}}\, du\, dy.$$

$$= \frac{1}{2^{\frac{n-2}{2}}\Gamma(\frac{n}{2})\sqrt{2\pi}} e^{-\frac{\mu^2}{2}} e^{-\frac{y^2}{2}(1+\frac{u^2}{n})} \sum_{i=0}^{\infty} \frac{(\mu u)^i\, y^{i+n}}{n^{\frac{i+1}{2}}\, i!}\, du\, dy \quad (2)$$

The marginal distribution of U is obtained by integrating (2) w.r.t. y from 0 to ∞.

$$\therefore \quad dF(u) = \frac{1}{2^{\frac{n-2}{2}}\Gamma(\frac{n}{2})\sqrt{2\pi}} e^{-\frac{\mu^2}{2}} \sum_{i=0}^{\infty} \frac{(\mu u)^i}{n^{\frac{i+1}{2}}\, i!}\, du \int_0^{\infty} e^{-\frac{y^2}{2}(1+\frac{u^2}{n})} y^{i+n}\, dy \quad (3)$$

Putting $y^2 = t \Rightarrow dy = \frac{dt}{2\sqrt{t}}$

$$\therefore \int_0^{\infty} e^{-\frac{y^2}{2}(1+\frac{u^2}{n})} y^{i+n}\, dy = \int_0^{\infty} e^{-\frac{t}{2}(1+\frac{u^2}{n})} t^{\frac{i+n}{2}} \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^{\infty} e^{-\frac{t}{2}(1+\frac{u^2}{n})} t^{\frac{i+n}{2}-\frac{1}{2}}\, dt$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{i+n+1}{2}\right)}{\left\{\frac{1}{2}(1+\frac{u^2}{n})\right\}^{\frac{i+n+1}{2}}} = \frac{2^{\frac{i+n+1}{2}-1}\,\Gamma\left(\frac{i+n+1}{2}\right)}{\left(1+\frac{u^2}{n}\right)^{\frac{i+n+1}{2}}}$$

Now

$$dF(u) = \frac{1}{2^{\frac{n-2}{2}}\Gamma(\frac{n}{2})\sqrt{2\pi}} e^{-\frac{\mu^2}{2}} \sum_{i=0}^{\infty} \frac{(\mu u)^i}{n^{\frac{i+1}{2}}\, i!}\, du \cdot \frac{2^{\frac{i+n-1}{2}}\,\Gamma\left(\frac{i+n+1}{2}\right)}{\left(1+\frac{u^2}{n}\right)^{\frac{i+n+1}{2}}}$$

$$= \frac{e^{-\frac{\mu^2}{2}}}{\Gamma(\frac{n}{2})\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(\mu u)^i}{i!} \frac{2^{i/2}}{\left(1+\frac{u^2}{n}\right)^{\frac{i+n+1}{2}}} \cdot \frac{\Gamma\left(\frac{i+n+1}{2}\right)}{n^{\frac{i+1}{2}}}\, du$$

$$-\infty < u < \infty$$

This is the distribution of non-central t statistic with n d.f. and non-centrality parameter μ. For $\mu = 0$, the non-central t distribution tends to central t distribution with n d.f as shown below:

$$dF(u) = \frac{1}{\Gamma(\frac{n}{2})\sqrt{\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{n^{1/2}\left(1+\frac{u^2}{n}\right)^{\frac{n+1}{2}}}\, du$$

$$= \frac{1}{B(\frac{1}{2},\frac{n}{2})} \frac{du}{\sqrt{n}\left(1+\frac{u^2}{n}\right)^{\frac{n+1}{2}}}, \quad -\infty < u < \infty$$

Moments of Non-Central t Statistic –

Suppose $U \sim$ Non-Central t disⁿ with n d.f. and non-centrality parameter μ, then the rth raw moment of the statistic U is

$$E(U^r) = E[X\sqrt{n}/Y]^r \quad , \text{ where } X \sim N(\mu, 1) \text{ and } Y^2 \sim \chi^2_{n \text{ d.f.}}$$

$$= E[X^r Y^{-r}]\, n^{r/2}$$

$$= n^{r/2} E(X)^r E(Y)^{-r} \quad \text{since } X \text{ and } Y \text{ are indep.}$$

Now $E(X)^r = \int_{-\infty}^{\infty} x^r \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2} dx$ Let $x-\mu = z$, $x = \mu + z$, $dx = dz$

$$= \int_{-\infty}^{\infty} \frac{(\mu+z)^r}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \mu^r \int_{-\infty}^{\infty} \left(1+\frac{z}{\mu}\right)^r e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \mu^r \int_{-\infty}^{\infty} \sum_j \binom{r}{j} \left(\frac{z}{\mu}\right)^j e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \sum_j \binom{r}{j} \mu^{r-j} \int_{-\infty}^{\infty} z^j e^{-\frac{1}{2}z^2} dz$$

Let $z^2 = \omega$, $2z\,dz = d\omega$ $\Rightarrow dz = \frac{d\omega}{2\sqrt{\omega}}$

$$= \frac{1}{\sqrt{2\pi}} \sum_j \binom{r}{j} \mu^{r-j} \int_0^{\infty} e^{-\omega/2} \omega^{\frac{j+1}{2}-1} \frac{d\omega}{2}$$

$$= \frac{1}{\sqrt{2\pi}} \sum_j \binom{r}{j} \mu^{r-j} \frac{\Gamma\left(\frac{j+1}{2}\right)}{2\left(\frac{1}{2}\right)^{\frac{j+1}{2}}} = \frac{1}{\sqrt{\pi}} \sum_{j}^{r} \binom{r}{j} \mu^{r-j} 2^{j/2} \Gamma\left(\frac{j+1}{2}\right)$$

and $E(Y)^{-r} = \int_0^{\infty} y^{-r} f(y)\, dy$

$$= \int_0^{\infty} y^{-r} \frac{1}{2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)} e^{-\frac{y^2}{2}} y^{n-1} dy$$

$$= \frac{1}{2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)} \int_0^{\infty} e^{-y^2/2} y^{n-r-1} dy$$

Let $y^2 = \omega$, $2y\,dy = d\omega$ $\Rightarrow dy = \frac{d\omega}{2\sqrt{\omega}}$

$$= \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} \int_0^{\infty} e^{-\omega/2} \omega^{\frac{n-r}{2}-1} d\omega$$

$$= \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\Gamma\left(\frac{n-r}{2}\right)}{\left(\frac{1}{2}\right)^{\frac{n-r}{2}}} = 2^{-r/2} \frac{\Gamma\left(\frac{n-r}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

Therefore $E(U^r) = \frac{\Gamma\left(\frac{n-r}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)} \sum_j \binom{r}{j} \mu^{r-j} 2^{(j-r)/2} \Gamma\left(\frac{j+1}{2}\right)$

$$= \frac{\Gamma\left(\frac{n-r}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)} \sum_{j}^{r} \binom{r}{j} \left(\mu/\sqrt{2}\right)^{r-j} \Gamma\left(\frac{j+1}{2}\right)$$