

## Measurement of Cyclic Variation :-

To find the cyclic components of a time-series, we use the method, generally called residual method. In this method 3 other components like trend, seasonal, irregular variations are removed from the time-series. But this method is not appropriate. So, we used some other methods.

(1) Harmonic Analysis :- Harmonic analysis is an application of Fourier series in time-series and is used to determine the cyclic variation. Fourier is a sum of sin's and cosin's f's. Thus, for a time-series  $y_t$  with period of oscillation  $\lambda$  can be represented as

$$\begin{aligned} \underset{t}{y}_t &= a_0 + \left[ a_1 \sin\left(\frac{2\pi}{\lambda} t\right) + a_2 \sin\left(\frac{2\pi}{\lambda} 2t\right) + \dots \right] \\ &\quad + \left[ b_1 \cos\left(\frac{2\pi}{\lambda} t\right) + b_2 \cos\left(\frac{2\pi}{\lambda} 2t\right) + \dots \right] \\ &= a_0 + \sum_{i=1}^{\infty} \left[ a_i \sin\left(\frac{2\pi}{\lambda} i t\right) + b_i \cos\left(\frac{2\pi}{\lambda} i t\right) \right] \end{aligned}$$

$$\text{where, } a_0 = \frac{1}{n} \sum_{t=1}^n y_t, \quad a_i = \frac{2}{n} \sum_{t=1}^n y_t \sin\left(\frac{2\pi}{\lambda} i t\right),$$

$$b_i = \frac{2}{n} \sum_{t=1}^n y_t \cos\left(\frac{2\pi}{\lambda} i t\right), \quad i=1, 2, \dots$$

$n$  be the no. of terms in time-series. After representing an oscillatory component of a time-series by harmonic or Fourier analysis, we observe that the hidden periodicity in the time-series represented by  $\lambda$  which is unknown. If we can estimate the value of  $\lambda$ , then we are able to find the values of other involved parameters (i.e. we can estimate cyclic variation). To estimate the value of hidden periodicity  $\lambda$ , we may use the periodogram analysis.

Periodogram Analysis :- In periodogram analysis, we assume that the trend and seasonal components of a time-series has been eliminated. In other words, the time-series  $y_t$  consist only 2 components

- (1) Periodic with period  $\lambda$  and amplitude  $a$
- (2) Random component  $\epsilon_i$ ,

which is uncorrelated with any cyclic movement for a long time-series. Thus, we can write

$$y_t = a \sin\left(\frac{2\pi}{\lambda} t\right) + \epsilon_i \quad \text{--- (1)}$$

where,

$$\text{Cov}(\varepsilon_i, \varepsilon_j) = 0$$

Now,

$$\text{Var}(\varepsilon_i) = \sigma^2$$

$$\text{Cov}(\varepsilon_i, \sin\left(\frac{2\pi}{\lambda}t\right)) = 0$$

$$\text{Cov}(\varepsilon_i, \cos\left(\frac{2\pi}{\lambda}t\right)) = 0$$

Now, let us define  $A = \frac{2}{n} \sum_{t=1}^n y_t \sin\left(\frac{2\pi}{\lambda}t\right)$  and

$$B = \frac{2}{n} \sum_{t=1}^n y_t \cos\left(\frac{2\pi}{\lambda}t\right) \quad (2)$$

~~where~~

$\mu$  is the same trial period further

define

$$S^2(\mu) = A^2 + B^2 \quad (3)$$

$S(\mu)$  is the intensity corresponding to trial period  $\mu$ . Now, using (1) and (2), we get

$$A = \frac{2}{n} \sum_{t=1}^n \left( a \sin \frac{2\pi t}{\lambda} + \varepsilon_i \right) \cos\left(2\pi\right)$$

Nov.	30.69	1.13	34.680
Dec.	31.14	1.36	42.350

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**2.6. Measurement of Cyclic Movement.** A crude method of measuring cyclic fluctuations ( $C_t$ ) of a time series ( $U_t$ ) is provided by the so-called Residual Approach, which essentially consists in eliminating the two components, viz., trend ( $T_t$ ) and seasonal variation ( $S_t$ ) and then removing the random component or irregular variation ( $R_t$ ) by averaging. Using the multiplicative model

$$U_t = T_t \cdot S_t \cdot R_t \cdot C_t$$

the steps involved in the calculation of  $C_t$  are :

- (i) Calculate trend value  $T_t$  by moving average method and seasonal index  $S_t$  (which is to be taken in fractional form and not in percentage form), preferably by a moving average method.
- (ii) Divide  $U_t$  by  $T_t \times S_t$ .
- (iii) The resulting value gives  $C_t$ , the cyclic component.

The use of moving average method of suitable period averages (smoothens) out the random component  $R_t$ . Since the method requires, for its success, a number of conditions to be satisfied, it is very seldom used.

**Example 2.13.** Obtain the indices of cyclical variation for the data in example 2.9.

**Solution.** Cyclical variations are obtained by eliminating trend and seasonal variations from the given data. The seasonal variations are eliminated by dividing every figure by seasonal index of the corresponding quarters, and subtracting 100 from the result so obtained.

## CALCULATION OF CYCLICAL VARIATIONS

Year	(Trend Eliminated figures $\div$ Seasonal Index) $\times 100 - y$				Cyclical Variations $(y - \bar{y})$			
	I Qt.	II Qt.	III Qt.	IV Qt.	I Qt.	II Qt.	III Qt.	IV Qt.
1970	118.6	111.7	105.3	105.3	18.6	11.7	5.3	5.3
1971	93.6	104.3	107.6	102.6	-6.4	4.3	7.6	2.6
1972	84.5	90.6	92.0	89.7	-15.5	-9.4	-8.0	-10.3
1973	92.4	97.4	95.8	96.7	-7.6	-2.6	-4.2	-3.3
1974	115.2	99.7	103.3	109.7	15.2	-0.3	3.3	9.7

Harmonic Analysis provides a sophisticated method of determining the cyclic component of a time series.

Harmonic Analysis. From mathematical analysis we know that any function,  $U_t$  under some very general conditions, can be represented by a Fourier series, viz., a series of sums of sine and cosine functions. Thus for a time series ( $U_t$ ) with period of oscillation  $\lambda$ , we have

$$U_t = a_0 + a_1 \sin \frac{2\pi t}{\lambda} + a_2 \sin \frac{2\pi}{\lambda} \cdot 2t + \dots$$

$$+ b_1 \cos \frac{2\pi}{\lambda} t + b_2 \cos \frac{2\pi}{\lambda} \cdot 2t + \dots$$

...(\*)

where  $a_i = \frac{2}{n} \sum_{t=1}^n U_t \sin \left( \frac{2\pi}{\lambda} it \right), (i=1, 2, \dots)$

$$b_i = \frac{2}{n} \sum_{t=1}^n U_t \cos \left( \frac{2\pi}{\lambda} \cdot it \right), (i=1, 2, \dots)$$

$$a_0 = \frac{1}{n} \sum_{t=1}^n U_t$$

where  $n$  is the number of terms in the time series. For instance, if the period of oscillation is 12 months and  $U_1, U_2, \dots, U_{12}$  is the series or average of series for a number of years, then the constants  $a_i$ 's and  $b_i$ 's are given by

$$a_0 = \frac{1}{12} \sum_{t=1}^{12} U_t$$

$$a_i = \frac{2}{12} \sum_{t=1}^{12} U_t \sin \left( \frac{2\pi}{12} it \right), (i=1, 2, \dots, 6)$$

$$b_i = \frac{2}{12} \sum_{t=1}^{12} U_t \cos \left( \frac{2\pi}{12} it \right), (i=1, 2, \dots, 5)$$

So far  $\lambda$  in (\*) is regarded as a known constant. Periodogram Analysis provides an elegant method of determining  $\lambda$ .

Let us consider a time series in which the trend and the seasonal component have been eliminated. In other words, the resulting series  $U_t$  consists of only two components, one periodic with period  $\lambda$ , (say), and amplitude 'a' and the other, the random component  $\epsilon_t$ , (say) which is uncorrelated with any cyclic movement, for long time at least. Thus

$$U_t = a \sin \frac{2\pi}{\lambda} t + \epsilon_t \quad \dots(2.42)$$

$$\left. \begin{aligned} \text{Cov}(\epsilon_i, \epsilon_j) &= 0, (i \neq j); \text{Var}(\epsilon_i) = \sigma^2 \\ \text{Cov}\left(\epsilon_t, \sin \frac{2\pi t}{\lambda}\right) &= 0; \text{Cov}\left(\epsilon_t, \cos \frac{2\pi t}{\lambda}\right) = 0 \end{aligned} \right\} \dots(2.43)$$

Let us consider

$$\left. \begin{aligned} A &= \frac{2}{n} \sum_{t=1}^n U_t \cos \frac{2\pi t}{\mu} \\ B &= \frac{2}{n} \sum_{t=1}^n U_t \sin \frac{2\pi t}{\mu} \end{aligned} \right\} \dots(2.44)$$

where  $\mu$  is arbitrary and define

$$S^2(\mu) = A^2 + B^2 \quad \dots(2.45)$$

which is known as intensity corresponding to the trial period  $\mu$ .

Substituting from (2.42) in (2.44) and using (2.43), we get

intensity - the quality of being intense  
intense - of extreme force, degree, strength

$$\begin{aligned} A &= \frac{2}{n} \sum_{t=1}^n \left( a \sin \frac{2\pi t}{\lambda} + \epsilon_t \right) \cos \frac{2\pi t}{\mu} \\ &= \frac{a}{n} \sum_{t=1}^n 2 \sin \frac{2\pi t}{\lambda} \cos \frac{2\pi t}{\mu} \\ &= \frac{a}{n} \sum_{t=1}^n 2 \sin \alpha t \cos \beta t, \end{aligned}$$

where  $\alpha = \frac{2\pi}{\lambda}$  and  $\beta = \frac{2\pi}{\mu}$ .

$$\therefore A = \frac{a}{n} \sum_{t=1}^n \left[ \sin(\alpha + \beta)t + \sin(\alpha - \beta)t \right] \quad \dots(2.46)$$

Let  $S = \sum_{t=1}^n \sin(\alpha + \beta)t$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$\mu A \sin B = \cos(A-B) - \cos(A+B)$   
 $\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{C-D}{2}$

Time Series

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$$\begin{aligned}
 &= S \cdot \sin \left( \frac{\alpha + \beta}{2} \right) = \frac{1}{2} \sum_{r=1}^n 2 \sin (\alpha + \beta)r \cdot \sin \left( \frac{\alpha + \beta}{2} \right) \\
 &= \frac{1}{2} \sum_{r=1}^n \left[ \cos \left\{ (\alpha + \beta)r - \frac{\alpha + \beta}{2} \right\} - \cos \left\{ (\alpha + \beta)r + \frac{\alpha + \beta}{2} \right\} \right] \\
 &= \frac{1}{2} \left[ \left\{ \cos \left( \frac{\alpha + \beta}{2} \right) - \cos \frac{3(\alpha + \beta)}{2} \right\} + \left\{ \cos \frac{3(\alpha + \beta)}{2} - \cos \frac{5(\alpha + \beta)}{2} \right\} \right. \\
 &\quad \left. + \dots + \left\{ \cos \frac{(2n-1)(\alpha + \beta)}{2} - \cos \frac{(2n+1)(\alpha + \beta)}{2} \right\} \right] \\
 &= \frac{1}{2} \left[ \cos \frac{\alpha + \beta}{2} - \cos \frac{(2n+1)(\alpha + \beta)}{2} \right] \\
 &= \sin \left\{ \frac{(n+1)(\alpha + \beta)}{2} \right\} \cdot \sin \frac{\alpha + \beta}{2} \\
 \Rightarrow S &= \sum_{r=1}^n \sin (\alpha + \beta)r = \frac{\sin \frac{n(\alpha + \beta)}{2} \cdot \sin \frac{(n+1)(\alpha + \beta)}{2}}{\sin \left( \frac{\alpha + \beta}{2} \right)}
 \end{aligned}$$

Similarly we can get the value of  $\sum_{r=1}^n \sin (\alpha - \beta)r$ .

Substituting in (2.46), we get

$$A = \frac{a}{n} \left[ \frac{\sin \frac{n(\alpha + \beta)}{2} \sin \frac{(n+1)(\alpha + \beta)}{2}}{\sin \left( \frac{\alpha + \beta}{2} \right)} + \frac{\sin \frac{n(\alpha - \beta)}{2} \sin \frac{(n+1)(\alpha - \beta)}{2}}{\sin \left( \frac{\alpha - \beta}{2} \right)} \right]$$

If  $\alpha \neq \beta$ , then  $A \rightarrow 0$  for a large  $n$ , since then the expression in bracket is bounded for all  $\alpha, \beta$  and  $n$ . However if  $\alpha \rightarrow \beta$ , i.e.,  $\alpha - \beta \rightarrow 0$ , then for large  $n$ , we get

$$\begin{aligned}
 A &= \lim_{n \rightarrow \infty} \frac{a}{n} \text{ [some finite quantity]} \\
 &\quad + \lim_{n \rightarrow \infty} \frac{a}{n} \cdot \frac{\sin \frac{n(\alpha - \beta)}{2} \sin \frac{(n+1)(\alpha - \beta)}{2}}{\sin \left( \frac{\alpha - \beta}{2} \right)} \\
 &= a \sin \frac{(n+1)(\alpha - \beta)}{2} \text{ for large } n, \left( \text{since } \lim_{\theta \rightarrow 0} \frac{\sin n\theta}{\sin \theta} = n \right)
 \end{aligned}$$

Thus for large  $n$ ,

- if  $\alpha \neq \beta \Rightarrow \lambda \neq \mu$ , then  $A \rightarrow 0$  and
- if  $\alpha \rightarrow \beta \Rightarrow \lambda \rightarrow \mu$ , then  $A \rightarrow a \sin \frac{(n+1)(\alpha - \beta)}{2}$

Similarly it can be shown that for large  $n$ ,

if  $\alpha \neq \beta \Rightarrow \lambda \neq \mu$ , then  $B \rightarrow 0$  and

if  $\alpha \rightarrow \beta \Rightarrow \lambda \rightarrow \mu$ , then  $B \rightarrow a \cos \frac{(n+1)(\alpha-\beta)}{2}$

Thus if the arbitrary number ' $\mu$ ' is exactly the period of oscillation ( $\lambda$ ) of the series, then

$$S^2(\mu) = a^2 \Rightarrow S(\mu) = a \quad \dots(2.47)$$

On the other hand if  $\lambda \neq \mu$ , then  $S(\mu) = 0$ . Thus  $S(\mu)$  remains small unless the trial period  $\mu$  approaches the true period  $\lambda$  in which case its value is equal to the amplitude ' $a$ '. This conclusion forms the basis of Periodogram Analysis, which is summed up as follows:

From given time series  $U_1, U_2, \dots, U_n$  calculate  $A$  and  $B$  as defined in (2.44) for different values of  $\mu$  from 0 to  $n$  and compute  $S(\mu)$  using (2.45). The graph obtained on plotting  $S(\mu)$  against  $\mu$  is known as periodogram. The values of  $\mu$  corresponding to the 'significant' maximum values of  $S$  provide the fundamental periods of oscillation, provided no  $\mu_i$  is a multiple of another  $\mu$ ; the corresponding values of  $S$  provide the corresponding amplitudes.

**Remark.** The obvious drawback of Harmonic Analysis lies in 'huge calculations'. If by drawing the graph of the time series we can guess the true periods of oscillation, it may be necessary to compute  $S$  for only those values of the trial period  $\mu$  which are in the neighbourhood of approximate (guessed) values.

EXERCISE 2.10