Date
7/4/2020

Subt Signal & System
Class - B.tech 2nd yr (ECE)
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Topic: Properties of Z transform.

3. **Scaling in the Z-domain**

(Multiplication by Exponential Sequence):

Statement: If $x(n) \overset{Z}{\longleftrightarrow} X(Z)$

then $a^n \cdot x(n) \overset{Z}{\longleftrightarrow} X\left(\frac{Z}{a}\right)$.

For any constant 'a' real or complex

Proof. Acc to definition of Z-transform

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \text{--- (1)}$$

$$\therefore Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot a^n z^{-n} = \sum_{n=-\infty}^{\infty} x(n) (a^{-1} z)^{-n}$$

$$\therefore Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} x(n) \cdot \left(\frac{z}{a}\right)^{-n} \quad \text{--- (2)}$$

Comparing eq (1) and (2)

$$Z\{a^n x(n)\} = X\left(\frac{z}{a}\right)$$

$$\therefore \quad a^n x(n) \xleftrightarrow{Z} X\left(\frac{z}{a}\right)$$

Ex-1 obtain Z-transform of $x(n) = a^n u(n)$ using scaling property.

Sol: here $u(n)$ is a unit step Its Z-transform

$$Z\{u(n)\} = \frac{z}{z-1} \quad \text{--- (1)}$$

According to scaling property we have

$$a^n x(n) \xleftrightarrow{Z} X\left(\frac{z}{a}\right)$$

Let $x(n) = u(n)$

$Z\{a^n x(n)\} = Z\{a^n u(n)\}$ is

obtained by replacing z by $\frac{z}{a}$ in eq (1)

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$$\therefore Z\left\{a^n u(n)\right\} = \frac{z/a}{z/a - 1}$$

$$\therefore Z\left\{a^n u(n)\right\} = \frac{z}{z-a}$$

Ex. 3.3.18 : Find Z-transform and sketch the ROC

$x(n) = (-1)^n 2^{-n} u(n)$

Soln. :

$$\text{Given } x(n) = (-1)^n 2^{-n} u(n)$$

$$\therefore x(n) = (-1)^n \frac{1}{(2)^n} u(n)$$

$$\therefore x(n) = \left(-\frac{1}{2}\right)^n u(n)$$

Here $u(n)$ is unit step and its Z-transform is given by,

$$Z\{u(n)\} = \frac{Z}{Z-1} \quad \text{ROC} : |Z| > 1$$

According to scaling property, we can write,

$$Z\{a^n u(n)\} = \frac{Z/a}{Z/a - 1} : \text{ROC} \left|\frac{Z}{a}\right| > 1$$

Observe Equation (1). Here a = – 1/2. Putting this value in Equation (3) we get,

$$Z\left\{\left(-\frac{1}{2}\right)^n u(n)\right\} = \frac{Z/(-1/2)}{\left(\frac{Z}{-1/2}\right) - 1} \quad : \text{ROC} |Z| > \left|\left(-\frac{1}{2}\right)\right|$$

$$= \frac{-2Z}{-2Z-1} \quad : \text{ROC} |Z| > \left|-\frac{1}{2}\right|$$

$$\therefore Z\left\{\left(-\frac{1}{2}\right)^n u(n)\right\} = \frac{2Z}{2Z+1} \quad : \text{ROC} |Z| > 1/2$$

ROC is exterior part of circle having radius r = 1/2

Ex. 3.3.19 : Determine Z-transform including ROC of the following.

$$x(n) = \left(\frac{1}{2}\right)^n \{u(n) - u(n-10)\}$$

Soln. : The given expression can be written as,

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(n-10)$$

According to linearity property we have,

$$X(Z) = a_1 X_1(Z) + a_2 X_2(Z) \quad ...(2)$$

Recall the standard Z-transform pair,

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \qquad ROC: |Z| > |\alpha| \quad ...(3)$$

Consider first term of Equation (1),

Let,
$$x_1(n) = \left(\frac{1}{2}\right)^n u(n)$$

Using Equation (3) we can write,

$$X_1(Z) = Z\left\{\left(\frac{1}{2}\right)^n u(n)\right\} = \frac{Z}{Z-1/2} = \frac{2Z}{2Z-1} \qquad ROC: |Z| > \frac{1}{2} \quad ...(4)$$

Now consider second term of Equation (1),

Let,
$$x_2(n) = \left(\frac{1}{2}\right)^n u(n-10) \quad ...(5)$$

First we will obtain Z-transform of u (n – 10). Recall the time shifting property.

$$x(n-k) \overset{Z}{\longleftrightarrow} Z^{-k} \cdot X(Z)$$

Now we have,
$$u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-1}\ ;\ ROC: |Z| > 1$$

$$\therefore \quad Z\{u(n-10)\} = Z^{-10}\left(\frac{Z}{Z-1}\right)\ ;\ ROC\ |Z| > 1$$

$$\therefore \quad Z\{u(n-10)\} = \frac{Z^{-9}}{Z-1} \quad ...(6)$$

According to scaling property we have,

$$Z\{a^n x(n)\} = X\left(\frac{Z}{a}\right)$$

That means we have to replace Z by $\frac{Z}{a}$.

In this case, $a = \frac{1}{2}$. Thus applying scaling property to Equation (6) we can write,

$$Z\left\{\left(\frac{1}{2}\right)^n u(n-10)\right\} = \frac{\left[\frac{Z}{(1/2)}\right]^{-9}}{\frac{Z}{1/2}-1} = \frac{(Z^{-9})\left(\frac{1}{1/2}\right)^{-9}}{\frac{Z}{1/2}-1}\ ; \qquad ROC: \left|\frac{Z}{1/2}\right| > 1$$

$$= \frac{Z^{-9} \cdot \left(\frac{1}{2}\right)^{+9}}{2Z-1}\ ; \qquad ROC: |Z| > \frac{1}{2}$$

$$X_1(Z) = Z\left\{\left(\frac{1}{2}\right)^9 u(n-10)\right\} = \frac{\left(\frac{1}{2}\right)^9}{Z^9(2Z-1)} \quad \text{ROC}: |Z| > \frac{1}{2}$$

Putting Equations (4) and (7) in Equation (2) we get,

$$X(Z) = \frac{2Z}{2Z-1} - \frac{\left(\frac{1}{2}\right)^9}{Z^9(2Z-1)}; \quad \text{ROC}: |Z| > \frac{1}{2}$$

$$\therefore X(Z) = \frac{2Z \times Z^9 - \left(\frac{1}{2}\right)^9}{Z^9(2Z-1)}; \quad \text{ROC}: |Z| > \frac{1}{2}$$

$$\therefore X(Z) = \frac{2Z^{10} - \left(\frac{1}{2}\right)^9}{Z^9(2Z-1)}; \quad \text{ROC}: |Z| > \frac{1}{2}$$

Thus ROC is exterior part of circle having radius r = 1/2.

Ex. 3.3.20 : Determine Z transform and draw ROC of the following signal $x(n) = (2)^{n+2} u(n-1)$. Is the signal causal ?

Soln. : The given signal is,

$$x(n) = (2)^{n+2} u(n-1)$$

$$\therefore x(n) = 2^n \cdot 2^2 u(n-1)$$

$$\therefore x(n) = 4 \cdot 2^n u(n-1)$$

First we will obtain Z transform of u (n – 1). We have,

$$u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-1}, \quad \text{ROC}: |Z| > 1$$

According to shifting property we have,

$$u(n-1) \overset{Z}{\longleftrightarrow} Z^{-1}\left[\frac{Z}{Z-1}\right], \quad \text{ROC}: |Z| > 1$$

$$\therefore Z\{u(n-1)\} = \frac{1}{Z-1}, \quad \text{ROC}: |Z| > 1$$

Now according to the scaling property, we have,

$$Z\{a^n u(n)\} = X\left(\frac{Z}{a}\right)$$

Here a = 2. Thus applying scaling property to Equation (2).

$$Z\{2^n u(n-1)\} = \frac{1}{\frac{Z}{2}-1}, \quad \text{ROC}: \left|\frac{Z}{2}\right| > 1$$

$$Z\{2^n u(n-1)\} = \frac{2}{Z-2}, \quad \text{ROC}: |Z| > 2$$

Thus for Equation (1) we can write,

$$Z\{4 \cdot 2^n u(n-1)\} = 4 \cdot \frac{2}{Z-2}; \qquad ROC : |Z| > 2$$

$$\therefore \quad X(Z) = \frac{8}{Z-2}, \qquad ROC : |Z| > 2$$

Here ROC is exterior part of circle having radius 2, as shown in Fig. P. 3.3.20. Since ROC is exterior part of circle; the given sequence is causal.

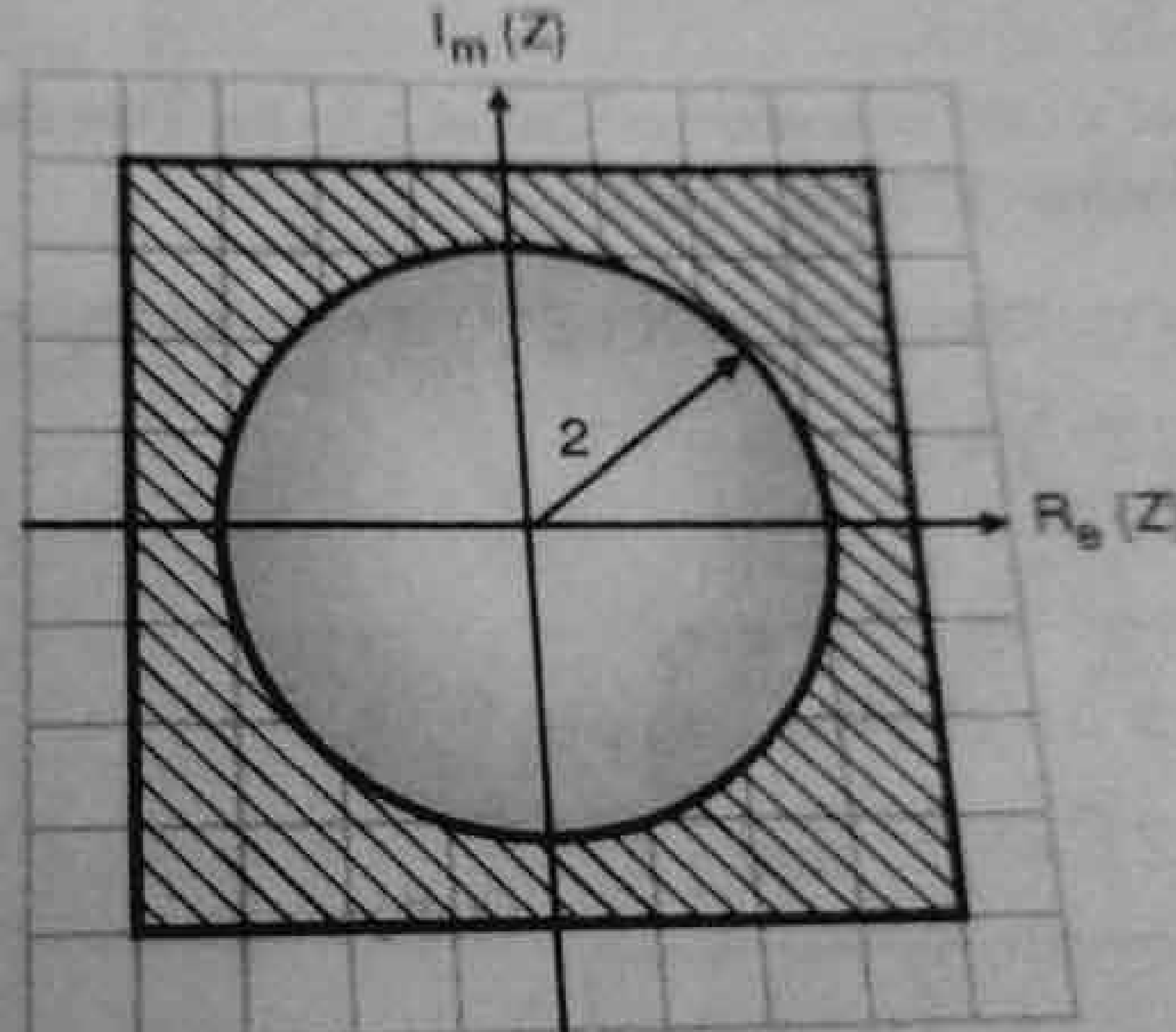

Fig. P. 3.3.20