4/4/2020

Subt Signal & System.

Class - 4th Sem. (ECE)

Date
Page

Faculty Name: Er. Mandeep Singh.

Unit- IV

"Topic - Z- Transform"

3. **Properties of Z- transform.**

1. **Linearity :-** It states that if

$$x(n) = a_1 x_1(n) + a_2 x_2(n)$$

and if $x_1(n) \xleftrightarrow{Z} X_1(Z)$ and

$x_2(n) \xleftrightarrow{Z} X_2(Z)$ then

where a_1 and a_2 are constants

(1) Linearity : It states that if

$$x(n) = a_1 x_1(n) + a_2 x_2(n)$$

and if $x_1(n) \xleftrightarrow{Z} X_1(Z)$ and

$x_2(n) \xleftrightarrow{Z} X_2(Z)$ then

where a_1 and a_2 are constants.

Proof : According to definition of Z-transform

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad \text{—— (1)}$$

here $x(n) = a_1 x_1(n) + a_2 x_2(n)$

$$\therefore \quad X(Z) = \sum_{n=-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)] Z^{-n} \quad \text{—— (2)}$$

writting two terms separately

$$X(z) = \sum_{n=-\infty}^{\infty} a_1 x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} a_2 x_2(n) z^{-n} \quad (3)$$

here a_1 and a_2 are constant. So we can take it outside the summation sign.

$$\therefore X(z) = a_1 \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

Comparing Eq. (3) with eq. (1) we get,

$$X(z) = a_1 X_1(z) + a_2 X_2(z)$$

<u>comment on ROC</u>: The combined ROC is the overlap or intersection of the individual ROC of $X_1(z)$ and $X_2(z)$.

Use <u>of this property</u>: We can express the given signal in the form of sum of two or more elementary sig. Then find their individual Z transform and add them.

Example: Determine Z-transform of
$$x(n) = (n+1)\,u(n)$$

Sol^n:

given function is
$$x(n) = (n+1)\,u(n)$$
$$\therefore \quad x(n) = n\,u(n) + u(n) \quad \text{--- (1)}$$

Let $x_1(n) = n\,u(n)$ and
$x_2(n) = u(n)$

$$\therefore \quad x(n) = x_1(n) + x_2(n) \quad \text{--- (2)}$$

here $n\,u(n)$ is ramp sequence we have standard Z-transform pair;

$$n\,u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{(Z-1)^2} \qquad \text{ROC: } |z| > 1$$

$$\therefore \quad X_1(Z) = Z\{n\,u(n)\} = \frac{Z}{(Z-1)^2}, \quad |z| > 1 \quad \text{--- (3)}$$

Now $u(n)$ is a unit step

$$u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-1}$$

Date ____
Page ____

$$X_2(z) = Z\{u(n)\} = \frac{z}{z-1}$$

Example 2 :- Find z-transform and ROC.

$$x(n) = a^n u(n) + \delta(n-5).$$

Sol:- The given function is;

$$x(n) = a^n u(n) + \delta(n-5). \quad \text{— (1)}$$

Let $x_1(n) = a^n u(n)$. — (2)

and $x_2(n) = \delta(n-5)$. — (3)

$$\therefore \; x(n) = x_1(n) + x_2(n) \quad \text{— (4)}$$

Firstly we will calculate z-transform of $x_1(n)$.

$$a^n u(n) \overset{z}{\longleftrightarrow} \frac{z}{z-a};$$

$$Z\{a^n u(n)\} = X_1(z) = \frac{z}{z-a}$$

Now we will calculate z-transform $x_2($

We have standard Z-transform pair

$$\delta(n-k) \xleftrightarrow{Z} Z^{-k}.$$

$$\therefore Z\{\delta(n-5)\} = X_2(Z) = Z^{-5}$$

from eq. (4) we can write.

$$X(Z) = X_1(Z) + X_2(Z).$$

$$X(Z) = \frac{Z}{Z+a} + Z^{-5}$$

combined ROC is $|Z| > |a|$.

Ex. 3.3.5 : Determine the Z-transform of : $x(n) = (\cos \omega_0 n)\, u(n)$. PTU - May 20

Soln. : According to Euler's identity we have,

$$\cos \theta = \frac{1}{2} e^{j\theta} + \frac{1}{2} e^{-j\theta}$$

$$\therefore \cos \omega_0 n = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$

$$\therefore x(n) = \frac{1}{2} e^{j\omega_0 n} u(n) + \frac{1}{2} e^{-j\omega_0 n} u(n)$$

Now we have the known form,

$$x(n) = a_1 x_1(n) + a_2 x_2(n)$$

Comparing Equations (1) and (2) we get,

$$x_1(n) = e^{j\omega_0 n}, \qquad \text{here } a_1 = \frac{1}{2}$$

$$\text{and } x_2(n) = e^{-j\omega_0 n}, \qquad \text{here } a_2 = \frac{1}{2}$$

Using Linearity property we get,

$$X(Z) = a_1 X_1(Z) + a_2 X_2(Z)$$

$$\text{Here } x_1(n) = e^{j\omega_0 n} u(n) = \left(e^{j\omega_0}\right)^n u(n)$$

Recall the standard Z-transform pair,

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \qquad \text{ROC is } |Z| > |\alpha|$$

$$\text{Here } \alpha = e^{j\omega_0} \text{ Thus we get,}$$

$$X_1(Z) = \frac{Z}{Z - e^{j\omega_0}} \qquad \text{ROC is } |Z| > |e^{j\omega_0}|$$

$$\text{Now } x_2(n) = e^{-j\omega_0 n} u(n) = \left(e^{-j\omega_0}\right)^n u(n)$$

$$\text{Here } \alpha = e^{-j\omega_0}$$

$$\therefore X_2(Z) = \frac{Z}{Z - e^{-j\omega_0}} \qquad \text{ROC is } |Z| > |e^{-j\omega_0}|$$

Scanned with CamScanner

Putting Equations (4) and (5) in Equation (3) we get,

$$X(Z) = a_1 \frac{Z}{Z - e^{j\omega_0}} + a_2 \frac{Z}{Z - e^{-j\omega_0}}$$

$$\text{But } a_1 = a_2 = \frac{1}{2}$$

$$\boxed{\therefore X(Z) = \frac{1}{2}\left[\frac{Z}{Z - e^{j\omega_0}} + \frac{Z}{Z - e^{-j\omega_0}}\right]}$$

ROC :

For $X_1(Z)$, ROC is $|Z| > |e^{j\omega_0}|$

$$\text{We have } e^{j\theta} = \cos\theta + j\sin\theta$$

$$\therefore e^{j\omega_0} = \cos\omega_0 + j\sin\omega_0$$

$$\text{Now } |e^{j\omega_0}| = \sqrt{\cos^2\omega_0 + \sin^2\omega_0} = 1$$

Thus ROC is $|Z| > 1$

Similarly for $X_2(Z)$, ROC is $|Z| > |e^{-j\omega_0 n}|$

$$\text{We have, } e^{-j\omega_0} = \cos\omega_0 - j\sin\omega_0$$

$$\text{Thus } |e^{-j\omega_0}| = \sqrt{\cos^2\omega_0 + \sin^2\omega_0} = 1$$

Thus ROC is $|Z| > 1$

So the combined ROC is $|Z| > 1$.

This is shown in Fig. P. 3.3.5.

This is a standard Z-transform pair.

We can further simplify Equation (6) as follows.

We have $e^{j\omega_0}$

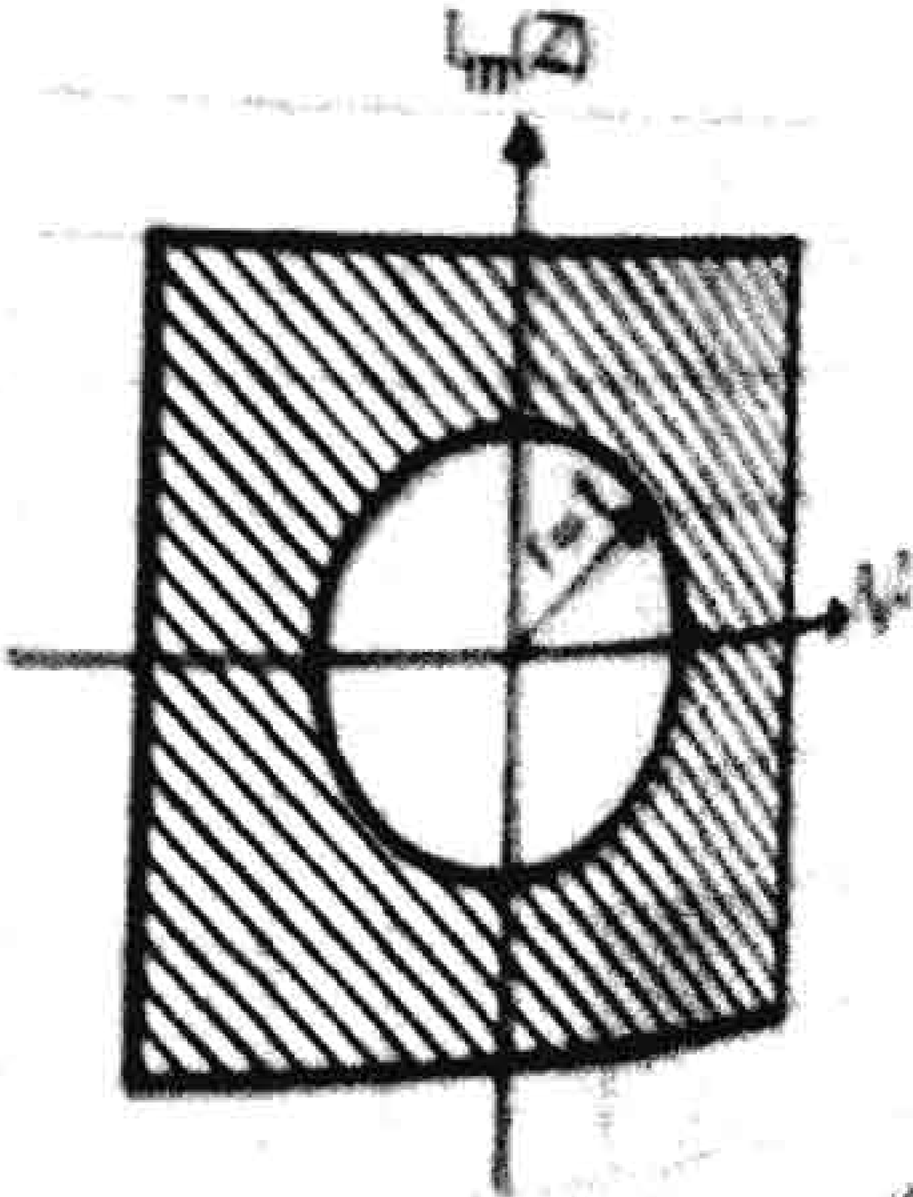

Fig. P. 3.3.5 : ROC of x(n) = (cos ...

ROC :

For $X_1(Z)$, ROC is $|Z| > |e^{j\omega_0}|$

$$\text{We have} \quad e^{j\theta} = \cos\theta + j\sin\theta$$

$$\therefore \quad e^{j\omega_0} = \cos\omega_0 + j\sin\omega_0$$

$$\text{Now} \quad |e^{j\omega_0}| = \sqrt{\cos^2\omega_0 + \sin^2\omega_0} = 1$$

Thus ROC is $|Z| > 1$

Similarly for $X_2(Z)$, ROC is $|Z| > |e^{-j\omega_0 n}|$

$$\text{We have, } e^{-j\omega_0} = \cos\omega_0 - j\sin\omega_0$$

$$\text{Thus } |e^{-j\omega_0}| = \sqrt{\cos^2\omega_0 + \sin^2\omega_0} = 1$$

Thus ROC is $|Z| > 1$

So the combined ROC is $|Z| > 1$.

This is shown in Fig. P. 3.3.5.

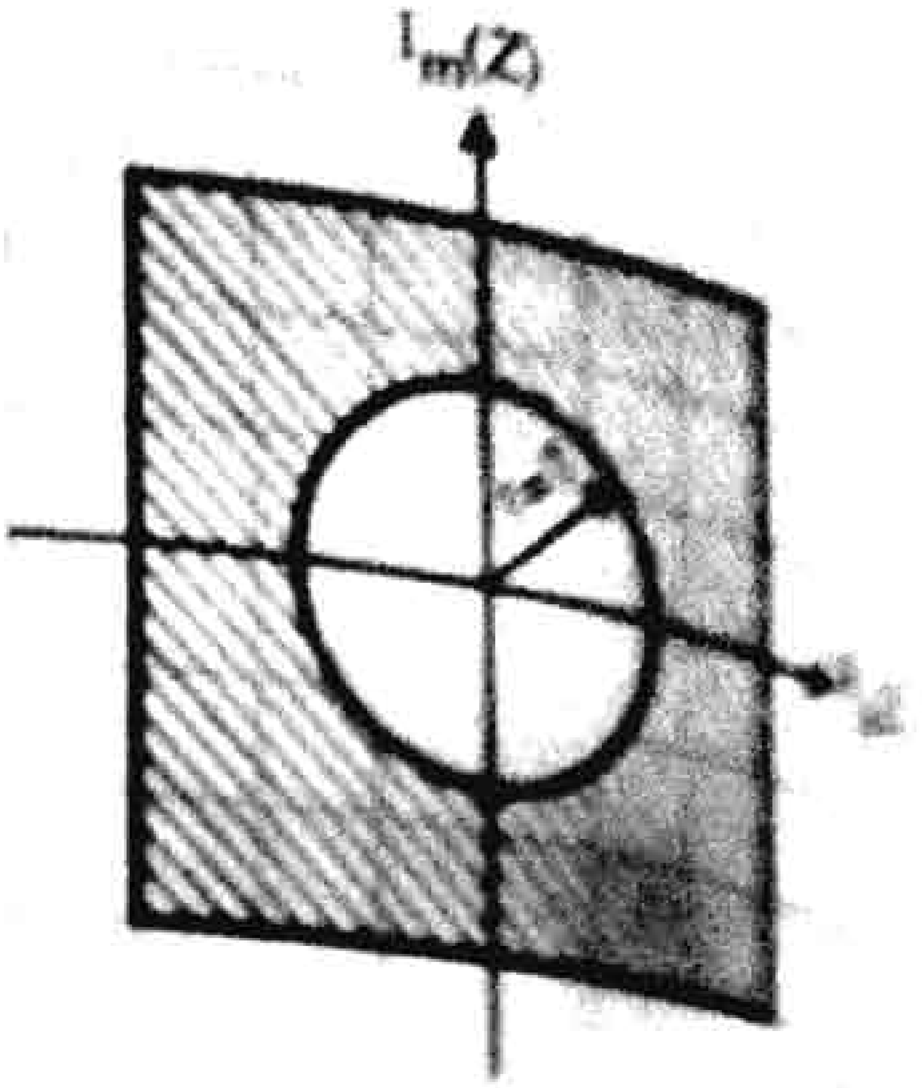

Fig. P. 3.3.5 : ROC of x (n) = (cos ...

This is a standard Z-transform pair.

We can further simplify Equation (6) as follows.

We have $e^{j\omega_0} = \cos\omega_0 + j\sin\omega_0$ and $e^{-j\omega_0} = \cos\omega_0 - j\sin\omega_0$

Putting these values in Equation (6) we get,

$$X(Z) = \frac{1}{2}\left[\frac{Z}{Z-(\cos\omega_0 + j\sin\omega_0)} + \frac{Z}{Z-(\cos\omega_0 - j\sin\omega_0)}\right.$$

Consider the first term inside the bracket,

$$\frac{Z}{Z-(\cos\omega_0 + j\sin\omega_0)} = \frac{Z}{Z-\cos\omega_0 - j\sin\omega_0}$$

Rationalizing the equation,

$$= \frac{Z\,(Z - \cos\omega_0 + j\sin\omega_0)}{(Z - \cos\omega_0 - j\sin\omega_0)\,(Z - \cos\omega_0 + j\sin\omega_0)}$$

$$= \frac{Z\,(Z - \cos\omega_0 + j\sin\omega_0)}{Z^2 - Z\cos\omega_0 + jZ\sin\omega_0 - Z\cos\omega_0 + \cos^2\omega_0 - j\sin\omega_0\cos\omega_0 - jZ\sin\omega_0 + j\sin\omega_0\cos\omega_0 - j^2\sin^2\omega_0}$$

$$= \frac{Z\,(Z - \cos\omega_0 + j\sin\omega_0)}{Z^2 - 2Z\cos\omega_0 + 1} \qquad \ldots(8)$$

...[Here $j^2 = -1$ and $\sin^2\omega_0 + \cos^2\omega_0 = 1$]

Now consider second term,

$$\frac{Z}{Z - (\cos\omega_0 - j\sin\omega_0)} = \frac{Z}{Z - \cos\omega_0 + j\sin\omega_0}$$

Rationalizing the equation,

$$= \frac{Z\,(Z - \cos\omega_0 - j\sin\omega_0)}{(Z - \cos\omega_0 + j\sin\omega_0)\,(Z - \cos\omega_0 - j\sin\omega_0)}$$

$$= \frac{Z\,(Z - \cos\omega_0 - j\sin\omega_0)}{Z^2 - 2Z\cos\omega_0 + 1} \qquad \ldots(9)$$

Putting Equations (8) and (9) in Equation (7) we get,

$$X(Z) = \frac{1}{2}\left[\frac{Z\,(Z - \cos\omega_0 + j\sin\omega_0)}{Z^2 - 2Z\cos\omega_0 + 1} + \frac{Z\,(Z - \cos\omega_0 - j\sin\omega_0)}{Z^2 - 2Z\cos\omega_0 + 1}\right]$$

$$\therefore\ X(Z) = \frac{1}{2}\left[\frac{Z^2 - Z\cos\omega_0 + jZ\sin\omega_0 + Z^2 - Z\cos\omega_0 - jZ\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}\right]$$

$$\therefore\ X(Z) = \frac{1}{2}\left[\frac{2Z^2 - 2Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}\right]$$

$$\therefore\ X(Z) = \frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}$$

Therefore the standard Z-transform identity is,

$$\boxed{(\cos\omega_0 n)\,u(n) \overset{Z}{\longleftrightarrow} \frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1},\quad \text{ROC is } |Z| > 1}$$

Ex. 3.3.6 : Determine Z-transform of :
$x(n) = \sin\omega_0 n\, u(n)$ **PTU**

Soln. : We can obtain Z-transform of $\sin\omega_0 n\, u(n)$ similar to the last problem using Eu... This problem can also be solved using another method, which is more simple.

We have ...

$$a^n u(n) \xleftrightarrow{Z} \frac{Z}{Z-a} \qquad \text{ROC is } |Z| > |a|$$

Let $a = e^{j\omega n}$. Thus we can write,

$$Z\{e^{j\omega n} u(n)\} = \frac{Z}{Z-e^{j\omega}} \qquad \text{ROC} : |Z| > |e^{j\omega_0}|$$

But we have,

$$e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n$$

Putting this value in Equation (3) we get,

$$Z\{\cos\omega_0 n + j\sin\omega_0 n\} = \frac{Z}{Z-(\cos\omega_0 + j\sin\omega_0)} = \frac{Z}{Z-\cos\omega_0 - j\sin\omega_0}$$

$$= \frac{Z(Z-\cos\omega_0 + j\sin\omega_0)}{(Z-\cos\omega_0 - j\sin\omega_0)(Z-\cos\omega_0 + j\sin\omega_0)}$$

$$\therefore Z\{\cos\omega_0 n + j\sin\omega_0 n\} = \frac{Z^2 - Z\cos\omega_0 + jZ\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}$$

$$\therefore Z\{\cos\omega_0 n\} + Z\{j\sin\omega_0 n\} = \frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} + j\frac{Z\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}$$

Comparing imaginary part we get,

$$Z\{j\sin\omega_0 n\} = \frac{jZ\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}$$

$$\therefore \quad Z\{\sin\omega_0 n\} = \frac{Z\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \qquad \text{ROC is } |Z| > |e^{j\omega_0}|$$

Similarly comparing real part we can obtain Z-transform of $\cos\omega_0 n$. Now ROC is $> |e^{j\omega_0}|$. But we know that $|e^{j\omega_0}| = 1$. Thus ROC is $|Z| > 1$. This ROC is same as shown in P. 3.3.5. This is a standard Z-transform pair.

$$\boxed{\therefore \quad \sin\omega_0 n\, u(n) \xleftrightarrow{Z} \frac{Z\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \qquad \text{ROC} : |Z| > 1}$$