Invariance Property of Suff. Statistic

Let $x_1, x_2, \dots x_n$ be i.i.d r.v's with pdf/pmf $f(x, \theta)$; $\theta \in \Theta$. Let T be a sufficient statistic for θ. If $\phi(T)$ is a one to one (single valued function). Then $\phi(T)$ is also a sufficient statistic for θ.

Completeness → A family of distributions $\{f(x,\theta); \theta \in \Theta\}$ is said to be complete if

$$E[h(X)] = 0$$
$$\Rightarrow h(X) = 0 \quad \forall x \in \Omega$$

where $h(x)$ is an arbitrary func^n.

Complete Statistic → A statistic is said to be complete if its family of distribution is complete.

Ex:

Let $X \sim B(1, p)$

$$p(x) = p^x (1-p)^{1-x} \quad ; x = 0, 1, \quad 0 < p < 1$$

$$E[h(X)] = \sum_{x=0}^{1} h(x)\, p(x)$$

$$= h(0)(1-p) + h(1) \cdot p$$

$= h(0) + p(h(1) - h(0))$

Now,

if

$E[h(X)] = 0$

$\Rightarrow h(0) + p(h(1) - h(0)) = 0$

$h(0) = 0$ & $h(1) - h(0) = 0$

$h(0) = 0 = h(1) = 0$

$\therefore h(x) = 0 \quad \forall x = 0, 1$

or $P[h(X) = 0] = 1$

$\therefore$ The family of $dist^n$ $B(1, p)$ is complete

Ex: Let $X \sim B(n, p)$

$p(x) = {}^nC_x p^x (1-p)^{1-x}$; $x = 0, 1, \ldots n$

$0 < p < 1$

$E[h(X)] = \sum_{x=0}^{n} h(x) p(x)$

= a polynomial of degree n in p

If $E[h(X)] = 0$

$\Rightarrow \sum_{x=0}^{n} h(x) p(x) = 0$

$\Rightarrow h(x) = 0 \quad \forall x = 0, 1, 2, \ldots n$

$\forall p \in (0, 1)$

$\therefore B(n, p)$ is complete.

Ex-1 Let $X_1, X_2, \ldots X_n$ be i.i.d Exp(θ)

$$f(x,\theta) = \frac{1}{\theta} e^{-x/\theta} ; \quad x, \theta > 0$$

$$E[h(X)] = \int_0^\infty h(x) \frac{1}{\theta} e^{-x/\theta} = 0$$

$$\Rightarrow \int_0^\infty h(x) e^{-x/\theta} = 0$$

$$\Rightarrow L[(h/x)] = 0$$

$$\Rightarrow h(x) = 0 \quad \forall x > 0$$

∴ The family of exp distⁿ Exp(θ) is complete

Ex- Let $X_1, X_2, \ldots, X_n$ be $G(\alpha, \beta)$

$$f(x) = \frac{\alpha^\beta}{\Gamma(\beta)} e^{-\alpha x} x^{\beta - 1} ; \quad x, \alpha, \beta > 0$$

Solⁿ: $E[h(X)] = \int_0^\infty h(x) f(x) dx$

$$= \int_0^\infty h(x) \cdot \frac{\alpha^\beta}{\Gamma(\beta)} e^{-\alpha x} x^{\beta-1} dx = 0$$

$$\Rightarrow \frac{\alpha^\beta}{\Gamma(\beta)} \int_0^\infty h(x) \cdot e^{-\alpha x} x^{\beta - 1} dx = 0$$

$$\Rightarrow \int_0^\infty h(x) \cdot e^{-\alpha x} x^{\beta-1} dx = 0$$

$$\Rightarrow \int^\infty h(x) \cdot x^{\beta-1} e^{-\alpha x} = 0$$

$$\Rightarrow L\left[h(x)\cdot x^{\beta-1}\right] = 0$$

$$\Rightarrow h(x)\cdot x^{\beta-1} = 0$$

$$\Rightarrow h(x) = 0 \quad \forall \quad x > 0$$

$\therefore$ Family of Gamma distⁿ is complete

Ex→ Let $X_1, X_2, \ldots X_n$ be $Exp(\theta)$
Then $Y = \Sigma X_i \sim G(\frac{1}{\theta}, n)$ which is a complete family of distⁿ.

Solⁿ:- $\therefore \Sigma X_i$ is a complete statistic

Minimum Variance Unbiased estimator

Let $x_1, x_2, \ldots, x_n$ be i.i.d r.v's with pdf / pmf

$f(x; \theta)$; $\theta \in \Theta$

An estimator T of θ, is said to be a (uniformly) minimum variance unbiased estimator of θ if

(i) T is unbiased for θ i.e;

$E(T) = \theta \quad \forall \; \theta \in \Theta$

(ii) $var(T) \le var(T') , \quad \forall \theta \in \Theta$

—

Thm → **CRAMER RAO INEQUALITY**

Regularity Conditions:-

1) The parametric space Θ is a subset of $\mathbb{R} = (-\infty, \infty)$

2) The sample space Ω i.e; range of x is independent θ.

4). Fisher's information $I(\theta) = E\left[\frac{\partial}{\partial \theta} \log f(\underset{\sim}{x}, \theta)\right]$

$= -n E\left[\frac{\partial^2}{\partial \theta^2} \log f(x, \theta)\right]$

= exists and is +ive

3) $\frac{\partial}{\partial \theta} \log f(\underset{\sim}{x}, \theta)$ exists $\forall\ \theta \in \Theta$

5) Differentiation under Integration sign is valid i.e.

$$\frac{\partial}{\partial \theta} \int_{\Omega} L(\underset{\sim}{x}, \theta)\, d\underset{\sim}{x} = \int_{\Omega} \frac{\partial}{\partial \theta} L(\underset{\sim}{x}, \theta)\, d\underset{\sim}{x}$$

Statement of C-R inequality:-

If T is an unbiased estimator of $\psi(\theta)$, then

$$Var(T) \geq \frac{[\psi'(\theta)]^2}{E\left[\frac{\partial}{\partial \theta} \log L(\underset{\sim}{x}, \theta)\right]^2}$$

$$= \frac{[\psi'(\theta)]^2}{-n\, E\left[\frac{\partial^2}{\partial \theta^2} \log f(\underset{\sim}{x}, \theta)\right]}$$

Ex→1) Let $x_1, x_2, \ldots, x_n$ be i.i.d r.v's exp(θ) with p.d.f

$$f(x; \theta) = \frac{1}{\theta} e^{-x/\theta} \quad ; x, \theta \in \Theta$$

Find the Minimum variance bound estimator and M.V.E of θ

Soln First we check the regularity conditions.

1) $\theta \in \Theta = (0, \infty) \subseteq \mathbb{R}$

2) $\Omega = (0, \infty)$ is ind. of θ.

3) $f(\underset{\sim}{x}, \theta)$

$= \frac{1}{\theta^n} e^{-\sum x_i/\theta}$

$\log f(\underset{\sim}{x}, \theta) = -n \log \theta - \frac{\sum x_i}{\theta}$

$\frac{\partial}{\partial \theta} \log f(\underset{\sim}{x}, \theta) = -\frac{n}{\theta} + \frac{\sum x_i}{\theta^2}$

exists $\forall \theta$

4) $f(x, \theta) = \frac{1}{\theta} e^{-x/\theta}$

$\log f(x, \theta) = -\log \theta - \frac{x}{\theta}$

$\frac{\partial}{\partial \theta} \log f(x, \theta) = -\frac{1}{\theta} + \frac{x}{\theta^2}$

$\frac{\partial^2}{\partial \theta^2} \log f(x, \theta) = \frac{1}{\theta^2} - \frac{2x}{\theta^3}$.

$E\left[\frac{\partial^2}{\partial \theta^2} \log f(x, \theta)\right] = \frac{1}{\theta^2} - \frac{2E(x)}{\theta^3}$

$= \frac{1}{\theta^2} - \frac{2\theta}{\theta^3} = \frac{1}{\theta^2} - \frac{2}{\theta^2}$

$= -\frac{1}{\theta^2}$

$\therefore \; -nE\left[\frac{\partial^2}{\partial\theta^2}\log f(x,\theta)\right] = \frac{n}{\theta^2}$ exists and is +ive

Since $f(x) = \frac{1}{\theta}e^{-x/\theta}$ is continuous in θ ...... Integrable over the range 0 to ∞

$\therefore$ differentiation and Integration is valid.

CR bound becomes $var(T) \geq \frac{1}{n/\theta^2}$

$\psi(\theta) = \theta$

$$V(T) \geq \frac{\theta^2}{n} \longrightarrow \text{Bound}$$

consider $T = \bar{x}$ so that $E(T) = E(\bar{x})$

$$V(T) = Var(\bar{x}) = \frac{\theta^2}{n}$$

$T = \bar{x}$ is MVBE for θ

Ex→ Let $x_1, x_2, \ldots, x_n$ be i.i.d $P(\theta)$ with p.m.f

$$f(x;\theta) = \frac{\theta^x e^{-\theta}}{x!} \; ; \; x = 0, 1, 2 \ldots, \; \theta > 0$$

find MVBE of θ

Solⁿ. i) $\theta \in \Theta = (0, \infty) \subset R$

ii) $\Omega = \{0, 1, 2 \ldots\}$ ind. of θ

(iii)

$$f(x) = \frac{e^{-\theta}\theta^{x}}{x!}$$

$$\log f(x) = -\theta + x\log\theta - \log x!$$

$$\frac{\partial \log f(x,\theta)}{\partial\theta} = -1 + \frac{x}{\theta} \quad \text{exists } \forall \theta.$$

(iv)

$$f(x,\theta) = \frac{e^{-\theta}\theta^{x}}{x!}$$

$$\frac{\partial^2}{\partial\theta^2}\log f(x,\theta) = -\frac{x}{\theta^2}$$

$$E\left[\frac{\partial^2}{\partial\theta^2}\log f(x,\theta)\right] = \frac{-\theta}{\theta^2} = \frac{-1}{\theta} \; ; \; E(X)=\theta$$

$$-n\left[\frac{\partial^2}{\partial\theta^2}\log f(x,\theta)\right] = \frac{n}{\theta}$$

$$= I(\theta) = FI \quad \text{exists f is +ve}$$

(iv) Note that, $f(x,\theta) = \frac{e^{-\theta}\theta^{x}}{x!}$

is a continuous func^n of θ and summable over $(0,1,2,\ldots)$

$\therefore$,

$$\frac{\partial}{\partial\theta}\sum f(x,\theta) = \sum\frac{\partial}{\partial\theta}f(x,\theta)$$

$\therefore$ for $\psi(\theta) = \theta$ be C-R bound is given by $V(T)$

$$V(T) \geq \frac{[\psi'(\theta)]^2}{I(\theta)} = \frac{1}{n/\theta} = \frac{\theta}{n}$$

Now, consider $T = \bar{X}$ so that

$$E(T) = E(\bar{x}) = \theta$$

and $Var(T) = Var(\bar{x}) = \frac{\theta}{n}$

$\therefore T = \bar{X}$ is MVBE of θ.

Ex-13 Let $X_1, X_2, \ldots X_n$ be iid r.v.'s following Binomial distⁿ $B(m, \theta)$ with pmf

$$f(x, \theta) = {}^{m}C_{x}\, \theta^{x} (1-\theta)^{m-x}, \quad x = 0, 1 \ldots m, \quad 0 < \theta < 1$$

find MVBE of θ.

① $\theta \in \Theta = (0, 1) \subset \mathbb{R}$

② $x \in \Omega = \{0, 1 \ldots n\}$ free from θ

③ $\log f(x) = \log {}^{m}C_{x} + x \log \theta + (m-x) \log (1-\theta)$

$$\frac{\partial \log f(x)}{\partial \theta} = \frac{x}{\theta} + \frac{(m-x)}{(1-\theta)} \times (-1)$$

$$= \frac{x}{\theta} - \frac{(m-x)}{(1-\theta)} \quad \text{exist}$$

④ $$\frac{\partial^2 \log f(x)}{\partial \theta^2} = \frac{-x}{\theta^2} - \frac{(m-x)}{(1-\theta)^2}$$

$$E\left[\frac{\partial^2 \log f(x)}{\partial \theta^2}\right] = \frac{-E(x)}{\theta^2} - \frac{E(m-x)}{(1-\theta)^2}$$

$$= \frac{-m\theta}{\theta^2} - \frac{m - m\theta}{(1-\theta)^2}$$

$$= \frac{-m\theta}{\theta^2} - \frac{m(1-\theta)}{(1-\theta)^2}$$

$$= \frac{-m + m\theta + m\theta}{\theta(1-\theta)}$$

$$= \frac{-m}{\theta} - \frac{m}{1-\theta}$$

$$-nE\left[\frac{\partial^2 \log f(x)}{\partial\theta^2}\right] = \left(\frac{m}{\theta} + \frac{m}{(1-\theta)}\right]n$$

$$= \left(\frac{1-\theta+\theta}{(1-\theta)\cdot\theta}\right) mn$$

$$I(\theta) = \frac{mn}{\theta(1-\theta)}$$

Rao Black well Theorem:-

Let $x_1, x_2, ---, x_n$ be iid r.v's with pdf / pmf $f(x, \theta)$ $\theta \in \Theta$, $x \in \Omega$. Let $Y = U(\underset{\sim}{X})$ be an unbiased estimator of θ. Let T be a sufficient statistic for θ.

Define $\phi(T) = E[Y/T=t]$

Then,

(i) $\phi(T)$ is unbiased for θ.

(ii) $var(\phi(T)) \leq var(Y)$.

Proof→ Pof of R.B Thm

Let

$f(y)$ - Marginal p.d.f of Y

$f(t)$ - Marginal pdf of t

$f(y,t)$ - Joint pdf of (y,t)

$h(Y|t) = \dfrac{f(y,t)}{f(t)}$ = conditional pdf of y given t

Now,

$$E[\phi(T)] = E_t[E_{y|t}(Y|T=t)]$$

$$= \int_t \left\{ \int_y y\, h(y|t)\, dy \right\} f(t)\, dt$$

$$= \int_t \left\{ \int_y y \frac{f(y,t)}{f(t)}\, dy \right\} f(t)\, dt$$

$$= \int_t \int_y y\, f(y,t)\, dy\, dt$$

$$= \int_y \int_t y\, f(y,t)\, dt\, dy$$

$$= \int_y y \left\{ \int_t f(y,t)\, dt \right\} dy$$

$$= \int_y y\, f(y)\, dy = E[Y] = \theta \quad \therefore E[\phi(T)] = \theta$$

$$\therefore \quad E[\phi(T)] = \theta$$

i.e. $\phi(T)$ is unbiased estimator of θ

Also consider, $Var(Y) = E(Y - E(Y))^2$

$$Var(Y) = E(Y-\theta)^2$$

$$= E\left[\{Y - \phi(T)\} + \{\phi(T) - \theta\}\right]^2$$

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$$= E[Y - \phi(T)]^2 + E[\phi(T) - \theta]^2 + 2E[\{Y - \phi(T)\}\{\phi(T) - \theta\}]$$

$$= E[Y - \phi(T)]^2 + var[\phi(T)] + 2E[\{Y - \phi(T)\}\{\phi(T) - \theta\}] \longrightarrow (1)$$

Now look at

$$E[\{Y - \phi(T)\}\{\phi(T) - \theta\}]$$

$$= \int_t \int_y [y - \phi(t)][\phi(t) - \theta] f(t, y) \, dy \, dt$$

$$= \int_t (\phi(t) - \theta)\left[\int_y (y - \phi(t)) \frac{f(t, y)}{f(t)} dy\right] f(t) dt$$

Note that

$$\int_y (y - \phi(t)) \frac{f(t, y)}{f(t)} dy$$

$$= \int_y y \, h(y|t) \, dy - \phi(t) \int_y h(y|t) \, dy$$

$$= \phi(t) - \phi(t) = 0$$

Hence,

$$E[\{Y - \phi(T)\}\{\phi(T) - \theta\}] = 0$$

$\therefore$ equ^n (1) becomes

$$\operatorname{Var}(Y) = E[Y - \phi(T)]^2 + \operatorname{Var}(\phi(T))$$

but $E[Y - \phi(T)]^2 \geq 0$

$\therefore$ $\operatorname{Var}(Y) \geq \operatorname{Var}(\phi(T))$ Hence Proved

'OR'

OR

Now,

$$E[(Y-\phi(T))(\phi(T)-\theta)]$$

$$= \int_t \int_y (y-\phi(t))(\phi(t)-\theta) f(y,t)\,dy\,dt$$

$$= \int_t \int_y (y-\phi(t))(\phi(t)-\theta)\, h(y|t) f(t)\,dy\,dt$$

$$= \int_t (\phi(t)-\theta) \left\{ \int_y (y-\phi(t))\, h(y|t)\,dy \right\} f(t)\,dt$$

$$= \int_t (\phi(t)-\theta) \{ E(y-\phi(T)) \} f(t)\,dt$$

$$= \int_t (\phi(t)-\theta) \{ E(Y/t) - \phi(T) \} f(t)\,dt$$

$$= \int_t (\phi(t)-\theta) \{ \phi(t) - \phi(t) \} f(t)\,dt$$

$$E[(y-\phi(T)](\phi(T)-\theta] = 0$$

So $\text{var}(Y) = E[Y-\phi(T)]^2 + \text{var}[\phi(T)]$

$= +\text{ive quantity} + \text{var}[\phi(t)$

Lehmann Scheffe: Thm

T is a complete sufficient statistic for θ and there exist unbiased estimator $Y = S(X)$ of θ. Then there is a unique UMVUE given by

$$\phi(T) = E[Y | T = t]$$

Pf: Let $\mathcal{U}$ be the class of all unbiased estimator of θ.
Let $Y_1, Y_2 \in \mathcal{U}$
such that

$$\phi_1(T) = E[Y_1 | T = t]$$
$$\phi_2(T) = E[Y_2 | T = t]$$

By R-B Thm both $\phi_1(T)$ and $\phi_2(T)$ are unbiased estimator for θ, so that

$$E[\phi_1(T) - \phi_2(T)] = 0 \quad \forall\, \theta \in \Theta$$
$$\Rightarrow \phi_1(T) - \phi_2(T) = 0$$

because T is complete

$\phi(T) = E[Y | T = t]$ is the unique UMVUE of θ $\forall$ $Y \in \mathcal{U}$

and

$$Var(\phi(T)) \le Var(Y) \quad \forall \; Y \in \mathcal{U}$$

Hence Proved

Ex: Let $X_1, X_2 - - X_n$ be iid r.v's ~ $P(\lambda)$ Then, find the UMVUE of $\psi(\lambda) = \xi$

Solution: $T = \Sigma X_i$ is suff. statistic in this case.

~ $P(\lambda)$ = complete family

$\therefore T = \Sigma X_i$ is complete suff.

Define

$$Y = \delta(\underset{\sim}{X}) = \begin{cases} 1 & \text{if } x_i = 0 \\ 0 & o.w. \end{cases}$$

So that,

$$E(Y) = E[\delta(\underset{\sim}{X})] = 1 \cdot P(X_1 = 0) + 0 \cdot P[X_1 \neq 0]$$

$$= P[X_1 = 0]$$

$$= \frac{e^{-\lambda}\lambda^0}{0!}$$

$$= e^{-\lambda}$$

i.e; $Y = \delta(\underset{\sim}{X})$ is unbiased for $e^{-\lambda}$

$\therefore$ By L-S Thm, the UMVUE of $e^{-\lambda}$ will be

$$\phi(T) = E[Y \mid T = t]$$

$$= \frac{P[X_1 = 0,\ T = t]}{P[T = t]}$$

$$= \frac{P[X_1 = 0,\ \sum_{i=2}^{n} x_i = t]}{P[T = t]}$$

$$= \frac{P[X_1 = 0]\ P[\sum_{i=2}^{n} X_i = t]}{P[T = t]}$$

$$= \frac{\frac{e^{-\lambda}\lambda^{0}}{0!} \cdot \frac{e^{-(n-1)\lambda}((n-1)\lambda)^{t}}{t!}}{\frac{e^{-n\lambda}(n\lambda)^{t}}{t!}}$$

$$= \left(\frac{n-1}{n}\right)^{t}$$

$$\left[\phi(t) = \left(1 - \frac{1}{n}\right)^{t} ; \quad t = 0, 1, 2 \ldots\right]$$

UMVUE of $e^{-\lambda}$

Ex→ Let $x_1, x_2, \ldots, x_n$ be iid exp(λ). Derive UMVUE of $\psi(\lambda) = e^{-2\lambda}$

Soln The pdf is

$$f(x, \lambda) = \lambda e^{-\lambda x} ; \quad x, \lambda > 0$$

The L.F is given by

$$L(\underset{\sim}{x}, \lambda) = \lambda^n e^{-\lambda \Sigma x_i}$$

By factorization thm $T = \Sigma x_i$ is a suff. Stat for λ. Also T is $G(n, \lambda)$ which is a complete family of $dist^n$

$\therefore T = \Sigma x_i$ is complete suff.

Define

$$Y = \delta(\underset{\sim}{X}) = \begin{cases} 1 & \text{if } x_1 > 2 \\ 0 & o.w. \end{cases}$$

So that

$$E(Y) = 1 \cdot P[x_1 > 2] + 0 \cdot P[x_1 \leq 2]$$

$$= P[X_1 > 2] = \int_2^\infty f(x) dx_1$$

$$= \int_2^\infty \lambda e^{-\lambda x_1} dx_1$$

$$= e^{-2\lambda}$$

$\therefore Y_1$ is an unbiased estimator of $\psi(\lambda) = e^{-2\lambda}$ By RB - LF Thm the UMVUE of $\psi(\lambda)$ is given by

is given by

$$\phi(t) = E[Y / T = t] = 1 \cdot P[x_1 > 2 / t] + 0 \cdot P[x_1 \leq 2 / t]$$

$$= P[x_1 > 2 / t]$$

$$= \int_{q}^{\infty} f(x_1|t)\, dt_1$$

where

$$f(x_1|t) = \frac{f(x_1;t)}{f(t)}$$

Define

$z = \sum_{i=2}^{n} x_i$ is $G(\lambda, n-1)$

Also x_1 is $Exp(\lambda)$

And, x_1 & z are independent

$\therefore$ the Joint pdf of x_1 & z is given by

$$f(x_1, z) = f(x_1)\, f(z)$$

$$= \lambda e^{-\lambda x_1} \cdot \frac{\lambda^{n-1}}{\Gamma(n-1)} z^{n-2} e^{-\lambda z} \quad ; x, z, \lambda > 0$$

$$= \frac{\lambda^n}{\Gamma(n-1)} z^{n-2} e^{-\lambda(z+x_1)}$$

Using the transformation

$u = x_1 \Rightarrow x_1 = u$

$y = x_1 + z \quad z = y - u$

Jacobian of the transformation

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial y} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$\therefore$ The joint p.d.f. of u & y is given by

$$f(u, y) = f(x, z)\,|J|$$

$$= \frac{\lambda^n}{\overline{|(n-1)}}(y-u)^{n-2} e^{-\lambda y};\ 0<u<y<\infty,\ \lambda>0$$

Now, replacing u by x_1, we get

$$f(x_1, y) = \frac{\lambda^n}{\overline{|(n-1)}}(y-x_1)^{n-2} e^{-\lambda y};$$

$$0<x_1<y<\infty$$
$$\lambda>0$$

Thus,

$$f(x_1|y) = \frac{f(x_1, y)}{f(y)}$$

$$= \frac{\dfrac{\lambda^n}{\overline{|(n-1)}}(y-x_1)^{n-2} e^{-\lambda y}}{\dfrac{\lambda^n}{\overline{|n}} y^{n-1} e^{-\lambda y}}$$

$$= (n-1)\frac{1}{y}\left(1-\frac{x_1}{y}\right)^{n-2};\ 0<x_1<y<\infty,\ \lambda>0$$

$$= (n-1)\frac{1}{t}\left(1-\frac{x_1}{t}\right);\ 0<x_1<t<\infty$$

$\therefore$ The UMVUE of $\psi(\lambda) = e^{-2\lambda}$ becomes,

$$\phi(t) = \int_2^{\infty} f(x_1 | t)\, dx_1$$

$$= \int_2^{\infty} (n-1)\frac{1}{t}\left(1-\frac{x_1}{t}\right)^{n-2} dx_1 \;;\; 0<x_1<t<\infty$$

$$= \int_2^{t} (n-1)\frac{1}{t}\left(1-\frac{x_1}{t}\right)^{n-2} dx_1$$

Put $1-\frac{x_1}{t} = \vartheta$

$-\frac{1}{t}\,dx_1 = d\vartheta$

$$= \int_0^{1-2/t} (n-1)\,\vartheta^{n-2}\,d\vartheta$$

$$= (n-1)\left.\frac{\vartheta^{n-1}}{n-1}\right|_0^{1-2/t}$$

$$= \left.\frac{\vartheta^{n-1}}{n-1}\right|_0^{1-2/t} = \left(1-\frac{2}{t}\right)^{n-1} \;;\; t>0$$

i.e; $\phi(t) = \left(1-\frac{2}{t}\right)^{n-1} \;;\; t>0$

Methods of estimation

Imp

1) Maximum likelihood estimator
2) Method of moments
3) Method of least squares
4) Method of minimum χ^2

1) Maximum likelihood estimation →

The principle of maximum likelihood consist in using an estimate $\hat{\theta}$ that value within the admissible range of θ i.e; within Θ, which maximises the likelihood funⁿ.

$$L(\underset{\sim}{x}, \theta) = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= L(\theta)$$

Thus the MLE estimator $\hat{\theta}$ satisfies.

$$L(\hat{\theta}) \geq L(\theta) \quad \forall \quad \theta \in \Theta$$

i.e; $L(\hat{\theta}) = \sup_{\theta \in \Theta} L(\theta)$

In practise it is usually easier to work with $\log L(\theta)$ which is a monotonically increasting funⁿ.

So $\hat{\theta}$ also satisfied

$$\log L(\hat{\theta}) \geq \log L(\theta) \quad \forall \quad \theta \in \Theta$$

i.e; $\log L(\hat{\theta}) = \sup_{\theta \in \Theta} \log L(\theta)$

Mostly we use differentiation Method to find MLE of θ wherever differentiation fails we use argument Method.

Ex: Let $x_1, x_2, \ldots, x_n$ be iid $\exp(\theta)$

The LF is

$$L(\underset{\sim}{x}, \theta) = \frac{1}{\theta^n} e^{-\sum x_i/\theta}$$

$$\log L = -n\log\theta - \frac{\sum x_i}{\theta}$$

$$\frac{\partial \log L}{\partial \theta} = -\frac{n}{\theta} + \frac{\sum x_i}{\theta^2} = 0$$

$$= \hat{\theta} = \bar{x}$$

$$\frac{\partial^2 \log L}{\partial \theta^2} = +\frac{n}{\theta^2} - \frac{2\sum x_i}{\theta^3}\Bigg|_{\hat{\theta} = \bar{x}}$$

$$= \frac{n}{\bar{x}^2} - \frac{2n\bar{x}}{\bar{x}^3}$$

$$= \frac{n}{\bar{x}^2}\left[1 - 2\right]$$

$$= -\frac{n}{\bar{x}^2} < 0$$

$\therefore$ $\hat{\theta} = \bar{x}$ is the MLE of θ.

Ex→ Let $x_1, x_2, \ldots, x_n$ be i.i.d $f(x, \theta)$

~~find the~~ MLE of $\theta = (\theta_1, \theta_2)$

1. $exp(\theta)$
2. $exp(\lambda)$
3. $N(\mu, \sigma^2)$ ← σ^2 known / μ / both unknown
4. $G(\alpha, \beta)$ ← α known / β known / both unknown
5. $Beta(a, b)$ ← a known / b known / both unknown
6. $B(1, \theta)$
7. $B(m, \theta)$; m known
8. $P(\theta)$
9. $NB(r, \theta)$; r known

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Ques 3. (a) Let $x_1, x_2, \ldots x_n$ be iid $G(\alpha, \beta)$

$$f(x, \alpha, \beta) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\alpha x}$$

$$L(\underset{\sim}{x}, \alpha, \beta) = \frac{\alpha^{n\beta}}{(\Gamma(\beta))^n} (\Pi x_i)^{\beta-1} e^{-\alpha \Sigma x_i}$$

$$\log L = n\beta \log\alpha - n \log \Gamma(\beta) + (\beta-1) \Sigma \log x_i - \alpha \Sigma x_i$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{n\beta}{\alpha} - \Sigma x_i = 0$$

$$\Rightarrow \boxed{\hat{\alpha} = \frac{\beta}{\bar{x}}} \longrightarrow (1)$$

$$\frac{\partial \log \beta}{\partial \beta} = n \log\alpha - n\left[\frac{\partial}{\partial \beta} \log \Gamma(\beta)\right] + \Sigma \log x_i = 0$$

diagamma func^n $\longrightarrow (2)$

Case I β is known

$$\hat{\alpha} = \frac{\beta}{\bar{x}}$$

Case II. α is known
Solve equ^n (2) Numerically to get $\hat{\beta}$

case II - Both are unknown

~~first~~ $\hat{\alpha} = \frac{\beta}{\bar{x}}$ in equ^n (2) and

Then solve it numerically to get $\hat{\beta}$. put this $\hat{\beta}$ in equ^n (1) to get $\hat{\alpha}$.

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8) $\exp(\lambda)$

$$L(\underset{\sim}{x}, \theta) = \lambda^n e^{-\lambda \sum x_i}$$

$$\log L = n \log \lambda - \lambda \sum x_i$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum x_i = 0$$

$$\frac{n}{\lambda} = n\bar{x}$$

$$\Rightarrow \lambda = \frac{1}{\bar{x}}$$

$$\frac{\partial^2 \log L}{\partial \lambda^2} = -\frac{n}{\lambda^2} < 0$$

$$\Rightarrow \left[\frac{\partial^2 \log L}{\partial \lambda^2}\right]_{\lambda = \frac{1}{\bar{x}}} = -n\bar{x}^2 < 0$$

Hence, MLE of $\hat{\lambda}$ is $1/\bar{x}$

(3) $X \sim N(\mu, \sigma^2)$

$$f(x; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} ; -\infty < x < \infty$$

Now

$$L(x; \mu, \sigma^2) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$\Rightarrow L(x; \mu, \sigma^2) = \left(\frac{1}{\sigma^2 2\pi}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

Taking log on both sides we get,

$$\log L = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 \rightarrow (1)$$

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(a) estimation of μ when σ^2 is known

$$\frac{\partial \log L}{\partial \mu} = 0$$

$$-0-0-\frac{1}{2\sigma^2} \times 2 \sum_{i=1}^{n} (x_i-\mu)(-1) = 0$$

$$\sum_{i=1}^{n} x_i - \mu \sum_{i=1}^{n} 1 = 0$$

$$\mu = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\Rightarrow \quad [\mu = \bar{x}]$$

$$\frac{\partial^2 \log L}{\partial \mu^2} = \frac{\partial}{\partial \mu}\left[\frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i-\mu)\right]$$

$$= \frac{1}{\sigma^2} \sum_{i=1}^{n} (0-1) \Rightarrow -\frac{n}{\sigma^2}$$

$$\text{So } \left[\frac{\partial^2 \log L}{\partial \mu^2}\right]_{\mu=\bar{x}} = -\frac{n}{\sigma^2} < 0$$

Hence MLE of μ is $\bar{x}$

i.e; $[\hat{\mu} = \bar{x}]$ Ans

(b) estimation of σ^2 when μ is known

$$\frac{\partial \log L}{\partial \sigma^2} = 0$$

$$0 - \frac{n}{2} \cdot \frac{1}{\sigma^2} - \frac{1}{2} \sum_{i=1}^{n} (x_i-\mu)^2 \times \frac{(-1)}{(\sigma^2)^2} = 0$$

$$\frac{1}{2(\sigma^2)^2} \sum_{i=1}^{n} (x_i-\mu)^2 = \frac{n}{2\sigma^2}$$

$$\Rightarrow \left[\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i-\mu)^2\right]$$

Now, $\frac{\partial^2 \log L}{\partial(\sigma^2)^2} = \frac{\partial}{\partial \sigma^2}\left[\frac{-n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(x_i-\mu)^2\right]$ (

$$= -\frac{n}{2}\frac{(-1)}{(\sigma^2)^2} + \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2 \times \frac{(-2)}{(\sigma^2)^3}$$

$$= \frac{n}{2(\sigma^2)^2} - \frac{1}{(\sigma^2)^3}\sum_{i=1}^{n}(x_i-\mu)^3$$

$$\left[\frac{\partial^2 \log L}{\partial(\sigma^2)^2}\right]_{\sigma^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2} = \frac{n\cdot n^2}{2\left\{\sum_{i=1}^{n}(x_i-\mu)^2\right\}^2} - \frac{1\times n^3\sum_{i=1}^{n}(x_i-\mu)^3}{\left\{\sum_{i=1}^{n}(x_i-\mu)^2\right\}^3}$$

$$= \frac{n^3}{2\left\{\sum_{i=1}^{n}(x_i-\mu)^2\right\}^2} - \frac{n^3}{\left\{\sum_{i=1}^{n}(x_i-\mu)^2\right\}^2}$$

$$= -\frac{n^3}{2\left\{\sum_{i=1}^{n}(x_i-\mu)^2\right\}^2} < 0$$

Hence, MLE of σ^2 is $\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$

$$\boxed{\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2}$$ Ans

(c) Simultaneous estimation of μ & σ^2

$\frac{\partial \log L}{\partial \mu} = 0$ gives

$$\Rightarrow \boxed{\hat{\mu} = \bar{x}}$$

and $\frac{\partial \log L}{\partial \sigma} = 0$ gives

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\mu})^2$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

$$\boxed{\hat{\sigma}^2 = s^2}$$

(viii) Given x is $P(\theta)$

$$f(x;\theta) = \frac{e^{-\theta}\theta^{x}}{x!} \quad ; x = 0, 1, 2 \ldots$$

Now.

$$L(x;\theta) = \frac{e^{-n\theta}\theta^{\sum x_i}}{\prod_{i=1}^{n} x_i!}$$

$$L(x;\theta) = (e^{-n\theta})\,\theta^{n\bar{x}} \cdot \frac{1}{\prod_{i=1}^{n} x_i!}$$

$$\log L(x;\theta) = -n\theta + n\bar{x}\log\theta - \sum_{i=1}^{n}\log x_i!$$

$$\frac{\partial \log L}{\partial \theta} = 0$$

$$\Rightarrow -n + \frac{n\bar{x}}{\theta} - 0 = 0$$

$$\boxed{\theta = \bar{x}}$$

Now,

$$\frac{\partial^2 \log L}{\partial \theta^2} = -0 + \frac{n\bar{x}(-1)}{\theta^2}$$

$$\left[\frac{\partial^2 \log L}{\partial \theta^2}\right]_{\theta=\bar{x}} = -\frac{n\bar{x}}{(\bar{x})^2} = \frac{-n}{\bar{x}} < 0$$

Hence MLE of θ is $\bar{x}$

$$\boxed{\hat{\theta} = \bar{x}}$$ Ans

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(5) Given $x \sim B(1, \theta)$

So, $f(x; \theta) = \theta^{x} (1-\theta)^{1-x}$; $x = 0, 1$

$\therefore L(x; p) = \theta^{\sum x_i} (1-\theta)^{n - \sum x_i}$

$L(x; p) = \theta^{n\bar{x}} (1-\theta)^{n - n\bar{x}}$

$\log L = n\bar{x} \log\theta + (n - n\bar{x}) \log(1-\theta)$

$$\frac{\partial \log L}{\partial \theta} = 0$$

$$\frac{\partial \log L}{\partial \theta} = \frac{n\bar{x}}{\theta} + \frac{(n - n\bar{x})}{(1-\theta)}(-1) = 0$$

$$\frac{\bar{x}}{\theta} = \frac{1-\bar{x}}{1-\theta}$$

$$\bar{x} - \bar{x}\theta = \theta - \theta\bar{x}$$

$$\Rightarrow \{\theta = \bar{x}\}$$

$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{n\bar{x}}{\theta^2} + \frac{(n - \bar{x}n)(-1)^3}{(1-\theta)^2}$$

$$\left[\frac{\partial^2 \log L}{\partial \theta^2}\right]_{\theta = \bar{x}} = \frac{-n\bar{x}}{\bar{x}^2} + \frac{(n - n\bar{x})(-1)^3}{(1-\bar{x})^2}$$

$$= -n\left[\frac{1}{\bar{x}} + \frac{(1-\bar{x})}{(1-\bar{x})^2}\right]$$

$$= -\frac{n}{\bar{x}(1-\bar{x})} < 0$$

Hence, MLE of θ is $\bar{x}$ $\Rightarrow$ $\boxed{\hat{\theta} = \bar{x}}$

(8) Given $x \sim NB(r, \theta)$; r is Known

$$P(X=x) = {}^{x+r-1}C_{r-1}\, \theta^{r} (1-\theta)^{x}; \quad x = 0, 1, \ldots$$

$$L(x; \theta) = \left({}^{x+r-1}C_{r-1}\right)^{n} \theta^{nr} (1-\theta)^{\sum_{i=1}^{n} x_i}$$

$$\log L = n \log\left({}^{x+r-1}C_{r-1}\right) + nr \log \theta + \sum_{i=1}^{n} x_i \log(1-\theta)$$

$$\log L = n \log\left({}^{x+r-1}C_{r-1}\right) + nr \log \theta + n\bar{x} \log(1-\theta)$$

$$\frac{\partial \log L}{\partial \theta} = 0$$

$$\frac{\partial \log L}{\partial \theta} = 0 + \frac{nr}{\theta} + \frac{n\bar{x}}{(1-\theta)}(-1) = 0$$

$$\frac{r}{\theta} = \frac{\bar{x}}{1-\theta}$$

$$r - r\theta = \bar{x}\theta$$

$$\Rightarrow r = \theta(r + \bar{x})$$

$$\left\{ \theta = \frac{r}{r+\bar{x}} \right\}$$

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$$\frac{\partial^2 \log L}{\partial \theta^2} = \frac{-nr}{\theta^2} + \frac{n\bar{x}(-1)^3}{(1-\theta)^2}$$

$$\left[\frac{\partial^2 \log L}{\partial \theta^2}\right]_{\theta = \frac{r}{r+\bar{x}}} = -\frac{nr(r+\bar{x})^2}{r^2} + \frac{n\bar{x}(-1)^3}{\left(1 - \frac{r}{r+\bar{x}}\right)^2}$$

$$= -\frac{n(r+\bar{x})^2}{r} - \frac{n\bar{x}(r+\bar{x})^2}{\bar{x}^2}$$

$$= -n(r+\bar{x})^2\left[\frac{1}{r} + \frac{1}{\bar{x}}\right]$$

$$= \frac{-n(r+\bar{x})^3}{r\bar{x}} < 0$$

Hence, MLE of θ is $\frac{r}{r+\bar{x}}$

i.e, $\left[\hat{\theta} = \frac{r}{r+\bar{x}}\right]$ Ans

Ques: Let $x_1, x_2, \dots, x_n$ be iid

$f(x;\theta) = \frac{1}{2\theta}$ $-\theta < x < \theta$

Soln.: $f(x;\theta) = \frac{1}{2\theta}$; $-\theta \le x \le \theta$

$= \frac{1}{2\theta} \, I_{(-\theta,\infty)}(x) \, I_{(-\infty,\theta)}(x)$

The pdf is given by

$$L(\underset{\sim}{X}) = \frac{1}{(2\theta)^n} \prod_{i=1}^{n} I_{(-\theta,\infty)}(x_i) \, I_{(-\infty,\theta)}(x_i)$$

$$= \frac{1}{(2\theta)^n} \, I_{(-\theta,\infty)}(x_{(1)}) \, I_{(-\infty,\theta)}(x_{(n)})$$

Note that $-\infty \quad -\theta \quad x_{(1)} \quad x_{(2)} \quad x_{(n)} \quad \theta \quad \infty$

Now

$x_{(1)} \ge -\theta$ and $x_{(n)} \le \theta$

$\Rightarrow -x_{(1)} \le \theta$ and $x_{(n)} \le \theta$

Do your self

Ques: Let $X_1\ X_2 \ldots x_n$ be iid $U(0,\theta)$

$$f(x;\theta) = \frac{1}{\theta} \;;\; 0<x<\theta$$

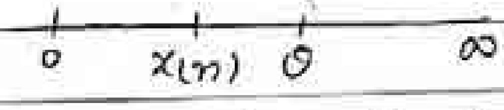
$$= \frac{1}{\theta} I_{(0,\theta)}(x) \;;\; x>0$$

$$L(\underset{\sim}{x},\theta) = \frac{1}{\theta^n} \prod_{i=1}^{n} I_{(0,\theta)}(x_i)$$

$$= \frac{1}{\theta^n} I_{(0,\theta)}(x_{(n)})$$

Note that

$$x_{(n)} \le \theta$$



$\therefore$ Max. value of $L(x,\theta)$ will be at minimum θ i.e; $\hat{\theta} = x_{(n)}$

$\therefore$ MLE of θ is $\hat{\theta} = x_{(n)}$

Ques: Let $X_1\ X_2 \ldots X_n$ be iid $\exp(\theta,1)$

$$f(x,\theta) = e^{-(x-\theta)} \;;\; 0 \le \theta \le x < \infty$$

$$= e^{-(x-\theta)} I_{(\theta,\infty)}(x) \;;\; 0 \le \theta \le x < \infty$$

$$L(\underset{\sim}{x},\theta) = e^{-\sum(x_i-\theta)} \prod_{i=1}^{n} I_{(\theta,\infty)}(x_i)$$

$$= e^{-\sum(x_{(i)}-\theta)} I_{(\theta,\infty)}(x_{(1)})$$

$$= e^{-\sum(x_{(i)}-x_{(1)}) + n(x_{(1)}-\theta)} I_{(\theta,\infty)} x_{(1)}$$

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Note that

$0 \quad x_{(1)} \quad \theta \quad \infty$

i.e; $x_{(1)} \leq \theta$ & To maximise $L(\underset{\sim}{x}, \theta)$ we minimize $(x_{(1)} - \theta)$

so MLE of θ is $\hat{\theta} = x_{(1)}$

Ex→ Let $x_1 x_2 --- x_n$ be iid Exp(0,1)

$$f(x, \theta) = \frac{1}{\theta} e^{-(x-\mu)/\theta} \; ; 0 \leq \mu \leq x < \infty$$

Solⁿ

$$= \frac{1}{\theta} e^{-(x-\mu)/\theta} \underset{(\mu, \theta)}{I} (x); \; 0 \leq x < \infty$$

$$L(\underset{\sim}{x}, \theta) = \frac{1}{\theta^n} e^{-\Sigma(x_i - \mu)/\theta} \prod_{i=1}^{n} \underset{(\mu,\infty)}{I}(x_i)$$

$$= \frac{1}{\theta^n} e^{-\Sigma(x_{(i)} - \mu)/\theta} \underset{(\mu,\infty)}{I} (x_{(1)})$$

$$= \frac{1}{\theta^n} e^{-\Sigma(x_{(i)} - x_{(1)}) + n(x_{(1)} - \mu)/\theta} \underset{(\mu,\infty)}{I} (x_{(1)}$$

Note that

$0 \quad x_{(1)} \quad \mu \quad \infty$

i.e; $x_{(1)} \le \mu$ and To Maximise $L(\underset{\sim}{x}, \mu)$ we minimize $(x_{(1)} - \mu)$

$\therefore$ MLE of μ is $\hat{\mu} = x_{(1)}$

Putting $\hat{\mu} = x_{(1)}$ in $L(\underset{\sim}{x}, \theta)$ we get,

$$L(\underset{\sim}{x}, \theta) = \frac{1}{\theta^n} e^{-\Sigma(x_{(i)} - x_{(1)})/\theta}$$

Taking log on both sides

$$\log L = -n \log \theta - \frac{1}{\theta} \Sigma[(x_{(i)} - x_{(1)})]$$

$$\frac{\partial \log L}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^2} \Sigma[(x_{(i)} - x_{(1)})] = 0$$

$$-\frac{n}{\theta} + \frac{1}{\theta^2} \Sigma[(x_{(i)} - x_{(1)})] = 0$$

$$\Rightarrow \frac{1}{\theta} \Sigma[x_{(i)} - x_{(1)}] = n$$

$$\Rightarrow \boxed{\hat{\theta} = \frac{1}{n} \Sigma(x_{(i)} - x_{(1)})}$$

$$\left[\frac{\partial^2 \log L}{\partial \theta^2}\right]_{\hat{\theta} = \frac{1}{n}\Sigma[x_{(i)} - x_{(1)}]} = \frac{n \times n^2}{[\Sigma[x_{(i)} - x_{(1)}]]^2} - \frac{2n^3 \Sigma[x_{(i)} - x_{(1)}]}{[\Sigma(x_{(i)} - x_{(1)})]^3}$$

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$$= \frac{n^3}{t^2} - \frac{2n^3}{t^2}$$

$$= -\frac{n^3}{t^2} < 0$$

where $t = \sum(x_{(i)} - x_{(1)})$

Hence MLE of θ is

$$\hat{\theta} = \sum[x_{(i)} - x_{(1)}]$$

Ex Let $x_1, x_2, \ldots, x_n$ be iid

$$f(x, \theta) = \frac{1}{\beta - \alpha}; \quad 0 < \alpha \leq x \leq \beta < \infty$$

Soln.

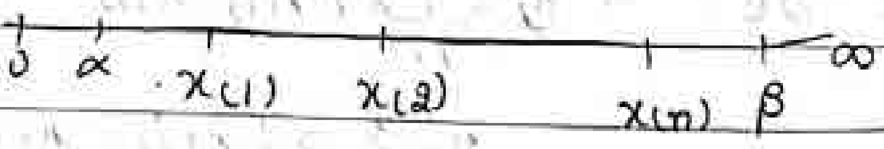
$$f(x, \alpha, \beta) = \frac{1}{\beta - \alpha}; \quad 0 < \alpha \leq x \leq \beta < \infty$$

$$= \frac{1}{\beta - \alpha} I_{(\alpha, \infty)}(x) \; I_{(0, \beta)}(x)$$

The LF is given by

$$L(\underset{\sim}{x}) = \frac{1}{(\beta - \alpha)^n} \prod_{i=1}^{n} I_{(\alpha, \infty)}(x_{(1)}) \; I_{(0, \beta)}(x_{(n)})$$

Note that



Here we want to do maximize to L.F. i.e; we want to do maximize $\frac{1}{(\beta-\alpha)}$

i.e; $\hat{\beta} = x_{(n)}$ The largest order statistic is MLE for β

$\hat{\alpha} = x_{(1)}$ The Smallest statistic is MLE for α

Ques. Let $x_1, x_2, \ldots x_n$ be iid

$f(x;\theta) = 1 \quad ; \quad 0 \le \theta - \frac{1}{2} \le x \le \theta + \frac{1}{2} < \infty$

Solution: $f(x) = 1$

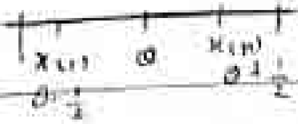
$$L(\underset{\sim}{x}) = \prod_{i=1}^{n} I_{(\theta-\frac{1}{2},\infty)}(x_i) \; I_{(0,\theta+\frac{1}{2})}(x_i)$$

$$= I_{(\theta-\frac{1}{2},\infty)}(x_{(1)}) \; I_{(0,\theta+\frac{1}{2})}(x_{(n)})$$

Lofo will be maximum when

$0 \le \theta - \frac{1}{2} \le x_{(1)} \; \ldots \; x_{(n)} \le \theta + \frac{1}{2} \le \infty$

$\theta \le x_{(1)} + \frac{1}{2} \quad , \quad \theta \ge x_{(n)} - \frac{1}{2}$



$x_{(n)} - \frac{1}{2} \le \theta \le x_{(1)} + \frac{1}{2}$

any values lies b/w $x_{(1)} + \frac{1}{2}$ & $x_{(n)} - \frac{1}{2}$

Ques: Let $x_1, x_2, \ldots, x_n$ be iid

$$f(x;\theta) = \frac{1}{2\theta} \quad -\theta < x < \infty$$

Solu^n: $f(x;\theta) = \frac{1}{2\theta}$; $-\theta \le x \le \infty$

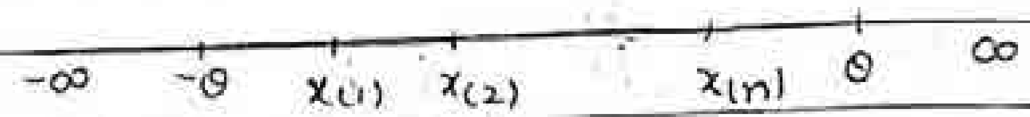
$$= \frac{1}{2\theta} I_{(-\theta,\infty)}(x) \, I_{(-\infty,\theta)}(x)$$

The LoF is given by

$$L(\underset{\sim}{x}) = \frac{1}{(2\theta)^n} \prod_{i=1}^{n} I_{(-\theta,\infty)}(x_i) \, I_{(-\infty,\theta)}(x_i)$$

$$= \frac{1}{(2\theta)^n} I_{(-\theta,\infty)}(x_{(1)}) \, I_{(-\infty,\theta)}(x_{(n)})$$

Note that



Now,

$$x_{(1)} \ge -\theta \quad \text{and} \quad x_{(n)} \le \theta$$

$$\Rightarrow -x_{(1)} \le \theta \quad \text{and} \quad x_{(n)} \le \theta$$

Do your self.

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2) Method of Moments

Let $x_1, x_2 \ldots x_n$ be a random Sample with pdf / pmf $f(x;\theta)$ $\theta \in \Theta$. Here θ may be vector value $\underset{\sim}{\theta} = (\theta_1, \theta_2 \ldots \theta_p)$

we define r^{th} popu^n moment as

$$\mu_r = E[X - \mu_1']^r \; ; \; r > 0$$

{r^{th} central moment}

where $\mu_1' = E(X)$
= 1^{st} raw moment
= popu^n Mean

Note that μ_2 = popu^n variance
μ_1 = zero

Similarly, we define
Sample mean = $m_1' = \bar{x}$
and r^{th} sample central moment to be $m_r = \frac{1}{n} \Sigma (x_i - \bar{x})^r$

In the Method of moments we equate K sample moments to the corresponding K popu^n moments we get The following equations

$$\mu_1' = \bar{x}$$

$\mu_r = m_r \; ; \; r = 2, 3, \ldots, K$

The soln of the above eqn's give moment estimator of the parameters.

Ex ⇒ Let $x_1, x_2 \ldots x_n$ be iid $f(x, \theta)$ find Moment estimator of θ

[redacted] i) $B(1, \theta) \Rightarrow \bar{x} = \hat{\theta}$

ii) $B(m, \theta) \Rightarrow \bar{x} = m\theta \Rightarrow \hat{\theta} = \bar{x}/m$

iii) $P(\theta) \Rightarrow \bar{x} = \theta \Rightarrow \hat{\theta} = \bar{x}$

iv) $Geo(\theta) \Rightarrow \bar{x} = \frac{\theta}{1-\theta} \Rightarrow \hat{\theta} = \frac{\bar{x}}{n + \bar{x}}$

(v) NB

(vi) $\exp(\theta) \Rightarrow \bar{x} = \theta$

(vii) $\exp(\lambda) \Rightarrow \bar{x} = \frac{1}{\lambda} \Rightarrow \hat{\lambda} = \frac{1}{\bar{x}}$

(viii) $U(0, \theta) \Rightarrow \bar{x} = \frac{\theta}{2} \Rightarrow \hat{\theta} = 2\bar{x}$

(ix) $N(\mu, \sigma^2) \Rightarrow \bar{x} = \mu, \quad s^2 = \sigma^2$

(x) $G(\alpha, \beta) \Rightarrow \bar{x} = \frac{\beta}{\alpha} \longrightarrow (1)$

$s^2 = \frac{\beta}{\alpha^2} \longrightarrow (2)$

Ex → Let $x_1, x_2, \ldots, x_n$ be iid $f(x, \theta)$ find moment estimator of θ

1) $B(1, \theta)$

First popuⁿ moment about origin is given by,

$$\mu_1' = \theta$$

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the B.D. Then the first sample moment about origin

$$m_1' = \frac{1}{n} \Sigma x_i = \bar{x}$$

Hence $\boxed{\hat{\theta} = \bar{x}}$