LAPLACE TRANSFORM

A transform is merely a mapping of a function from one space to another. While it may be very difficult to solve certain equations directly for a particular function of interest, it is often easier to solve a corresponding equation in terms of a transform of the function and then invert the transform to obtain the function. One particular transform that is quite useful for solving some types of differential equations as well as certain integral equations is the Laplace transform. Here we shall mention some of the important properties of the Laplace Transform.

Definition – Let $f(t)$ be a function of a positive real variable t. Then the Laplace transform (L.T.) or exponential transform of $f(t)$ is defined by

$$f^*(s) = \int_0^\infty e^{-st} f(t)\, dt = f^e(s)$$

for the range of value of s for which the integral exist. We shall write

$$f^*(s) = L\{f(t)\}$$

to denote the Laplace transform of $f(t)$

Some Important Properties of L.T. –

(i) Linearity Property – If c_i's are constants and

$$L\{f_i(t)\} = f_i^*(s), \quad i = 1, 2, \cdots, n$$

then $L[c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t)] = c_1 f_1^*(s) + c_2 f_2^*(s) + \cdots + c_n f_n^*(s)$

(ii) L.T. of derivatives –

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt$$

$$= \int_0^\infty e^{-st}\, d f(t)$$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= s f^*(s) - f(0)$$

$$= s L[f(t)] - f(0) \qquad (1)$$

Now $L[f''(t)] = s L[f'(t)] - f'(0)$

$$= s[s f^*(s) - f(0)] - f'(0)$$

$$= s^2 f^*(s) - s f(0) - f'(0)$$

Similarly,

$$L[f'''(t)] = s L[f''(t)] - f''(0)$$

$$= s[s^2 f^*(s) - s f(0) - f'(0)] - f''(0)$$

$$= s^3 f^*(s) - s^2 f(0) - s f'(0) - f''(0)$$

In general

$$L[f^{(n)}(t)] = s^n f^*(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

This property of the Laplace transform is often used in solving linear differential equations as well as linear differential difference equations.

(iii) LT of Integrals –

$$L\left[\int_0^t f(x)\,dx\right] = \int_0^\infty \underset{II}{e^{-st}} \underbrace{\left(\int_0^t f(x)\,dx\right)}_{I} dt$$

$$= -\left(\frac{e^{-st}}{s}\right) \int_0^t f(x)\,dx \Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \frac{1}{s} f^*(s)$$

(iv) Change of Origin property – Suppose that

$$g(t) = \begin{cases} 0 & , t < u \\ f(t-u) & , t \geq u \end{cases}$$

Then $L[g(t)] = \int_0^\infty e^{-st} g(t)\,dt = \int_u^\infty e^{-st} f(t-u)\,dt$

$$= \int_0^\infty e^{-s(u+x)} f(x)\,dx$$

$$= e^{-su} f^*(s)$$

Thus, as the graph of the function $f(t)$ is shifted to the right by an amount u, its Laplace transform is multiplied by e^{-su}.

(V) Change of scale property —

$$L[f(at)] = \int_0^\infty e^{-st} f(at)\,dt$$

$$= \int_0^\infty e^{-sx/a} f(x) \frac{dx}{a}$$

$$= \frac{1}{a} f^*\left(\frac{s}{a}\right)$$

(VI) The displacement theorem — Suppose that a function $f(t)$ is multiplied by $e^{-\lambda t}$. Then

$$L[e^{-\lambda t} f(t)] = \int_0^\infty e^{-st} e^{-\lambda t} f(t)\,dt = f^*(s+\lambda)$$

i.e. multiplication of a function by $e^{-\lambda t}$ increases the argument of its Laplace transform by λ.

(VII) Limit Property — (Initial Value Theorem)

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s f^*(s)$$

Proof — We have

$$\lim_{s\to 0} \int_0^\infty e^{-st} f'(t)\,dt = \lim_{s\to 0} s f^*(s) - f(0)$$

or $\int_0^\infty f'(t)\,dt = \lim_{s\to 0} s f^*(s) - f(0)$

or $f(\infty) - f(0) = \lim_{s\to 0} s f^*(s) - f(0)$

or $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s f^*(s)$ proved.

(Final Value Theorem)

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} s f^*(s)$$

Proof - We have

$$\lim_{s\to 0}\int_0^\infty e^{-st} f'(t)\,dt = \lim_{s\to\infty} s f^*(s) - f(0)$$

or $$0 = \lim_{s\to\infty} s f^*(s) - f(0)$$

or $$\lim_{s\to\infty} s f^*(s) = \lim_{t\to 0} f(t) \quad \text{Proved.}$$

(VIII) L.T. of a Convolution - The integral

$$\int_0^t f(t-u)\,g(u)\,du = f(t) \circledcirc g(t)$$

is called the ordinary convolution of the function $f(t)$ and $g(t)$. The L.T. of convolution is

$$L[f(t)\circledcirc g(t)] = \int_0^\infty e^{-st}\left(\int_0^t f(t-u)g(u)\,du\right)dt \qquad 0<u<t<\infty$$

$$= \int_0^\infty g(u)\,du \int_u^\infty e^{-st} f(t-u)\,dt, \text{ on change of limit of Integration.}$$

$$= \int_0^\infty g(u)\,du \int_0^\infty e^{-s(u+v)} f(v)\,dv$$

$$= \int_0^\infty e^{-su} g(u)\,du \int_0^\infty e^{-sv} f(v)\,dv$$

$$= g^*(s)\cdot f^*(s)$$

Thus, the L.T. of the convolution of two functions is the product of their Laplace transform. This is a useful property, for example, in solving integral equations containing integrals of the convolution type. Also it is very useful to obtain the inverse Laplace transform of a function.

Note that, if $f(t)$ and $g(t)$ are the p.d.f.'s of two indep. r.v.s. X and Y, then the p.d.f. of their sum is given by $f(t)\circledcirc g(t)$.