<u>Inverse Laplace Transform</u> — If $f^*(s)$ denotes the Laplace transform of $f(t)$, then $f(t)$ is called the inverse Laplace transform of $f^*(s)$ i.e

$$\text{if } L\{f(t)\} = f^*(s)$$

$$\text{then } L^{-1}\{f^*(s)\} = f(t)$$

In particular

$$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}, \quad L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$$

<u>Properties of Inverse Laplace Transform</u> —

[1] <u>Linearity Property</u> — If $f_i(t)$ is the inverse L.T. of $f_i^*(s)$, then for constants $c_1, c_2, \ldots$

$$L^{-1}\left\{\sum_{i=1}^{n} c_i f_i^*(s)\right\} = \sum_{i=1}^{n} c_i L^{-1}\{f_i^*(s)\} = \sum_{i=1}^{n} c_i f_i(t)$$

(ii) <u>Inverse L.T. of derivatives</u> – If $L^{-1}\{f^*(s)\} = f(t)$

then

$$L^{-1}\left\{\frac{d^n}{ds^n} f^*(s)\right\} = (-1)^n t^n f(t)$$

<u>Proof</u> – We have

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} f^*(s)$$

Therefore $L^{-1}\left\{\frac{d^n}{ds^n} f^*(s)\right\} = (-1)^n t^n f(t)$

(iii) <u>Inverse L.T. of Integrals</u> – If $L^{-1}\{f^*(s)\} = f(t)$

then $L^{-1}\left[\int_s^\infty f^*(u)du\right] = \frac{f(t)}{t}$

<u>Proof</u> – Since we have

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f^*(u)du \quad , \text{provided } \lim_{t \to 0} \frac{f(t)}{t} \text{ exists}$$

$$\therefore L^{-1}\left\{\int_s^\infty f^*(u)du\right\} = \frac{f(t)}{t}$$

(IV) <u>Change of Origin property</u> – If $L^{-1}\{f^*(s)\} = f(t)$

then

$$L^{-1}\{f^*(s-a)\} = e^{at}.f(t) \quad , s>a$$

<u>Proof</u> – We have $f^*(s) = \int_0^\infty e^{-st} f(t)dt$

$$\therefore \quad f^*(s-a) = \int_0^\infty e^{-(s-a)t} f(t)dt$$

$$= \int_0^\infty e^{-st} e^{at} f(t)dt = L\{e^{at} f(t)\}$$

So, $L^{-1}\{f^*(s-a)\} = e^{at} f(t)$

(v) Change of scale property — If $L^{-1}\{f^*(s)\} = f(t)$ then

$$L^{-1}\{f^*(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$$

Proof — We have $f^*(s) = \int_0^\infty e^{-st} f(t)\,dt$

Now $$f^*(as) = \int_0^\infty e^{-ast} f(t)\,dt$$

$$= \int_0^\infty e^{-sx} f\left(\frac{x}{a}\right) \frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-st} f\left(\frac{t}{a}\right) dt = \frac{1}{a} L\left\{f\left(\frac{t}{a}\right)\right\}$$

Therefore, $L^{-1}\{f^*(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$ Proved

(vi) The displacement theorem — If $L^{-1}\{f^*(s)\} = f(t)$ then

$$L^{-1}\{e^{-\lambda s} f^*(s)\} = \begin{cases} f(t-\lambda) & , t>\lambda \\ 0 & , t<\lambda \end{cases}$$

Proof — We have $f^*(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$\therefore \quad e^{-\lambda s} f^*(s) = \int_0^\infty e^{-(t+\lambda)s} f(t)\,dt$$

$$= \int_\lambda^\infty e^{-xs} f(x-\lambda)\,dx$$

$$= \int_\lambda^\infty e^{-st} f(t-\lambda)\,dt$$

$$= \int_0^\lambda e^{-st}\cdot 0\,dt + \int_\lambda^\infty e^{-st} f(t-\lambda)\,dt$$

$$\therefore L^{-1}\{e^{-\lambda s} f^*(s)\} = \begin{cases} f(t-\lambda) & , t>\lambda \\ 0 & , t<\lambda \end{cases}$$ Proved

<u>Partial Fraction Technique</u> – This technique is used to find the inverse Laplace transforms of a rational function such as $\frac{G(s)}{Q(s)}$, where $G(s)$ and $Q(s)$ are the polynomials in s and the degree of $G(s)$ is less than that of $Q(s)$. The following exercises will show how the theory of partial fractions can be used in finding inverse Laplace transform.

<u>Exercise-1</u> Find $L^{-1}\left\{\frac{s+2}{(s-4)(s-6)}\right\}$

<u>Solution</u> – Here the degree of $G(s) = s+2$ is one and that of $Q(s) = (s-4)(s-6)$ is two. So we can apply this technique.

Writing

$$\frac{s+2}{(s-4)(s-6)} \equiv \frac{A}{s-4} + \frac{B}{s-6} \qquad (1)$$

Multiplying both sides of (1) by $(s-4)$ and putting $s=4$, we get

$$-\frac{6}{2} = A \Rightarrow A = -3$$

Now again multiplying both sides of (1) by $(s-6)$ and putting $s=6$, we get

$$\frac{8}{2} = B \Rightarrow B = 4$$

Therefore,

$$\frac{G(s)}{Q(s)} = -\frac{3}{s-4} + \frac{4}{s-6}$$

Now using the table of Laplace transform, we get

$$L^{-1}\left\{\frac{G(s)}{Q(s)}\right] = -3\,L^{-1}\left[\frac{1}{s-4}\right] + 4\,L^{-1}\left[\frac{1}{s-6}\right]$$

$$= -3e^{4t} + 4e^{6t}$$

Exercise-2 Find $L^{-1}\left\{\frac{s+1}{s(s+2)}\right\}$

Solution – Writing

$$\frac{s+1}{s(s+2)} \equiv \frac{A}{s} + \frac{B}{s+2} \qquad (1)$$

To obtain value of A, we multiply both sides of (1) by s and put $s=0$, so that

$$\frac{1}{2} = A \Rightarrow A = 1/2$$

Now to obtain value of B, we multiply both sides of (1) by $s+2$ and put $s=-2$, we get

$$\frac{1}{2} = B \Rightarrow B = 1/2$$

Therefore,

$$\frac{s+1}{s(s+2)} = \frac{1}{2s} + \frac{1}{2(s+2)}$$

Now.

$$L^{-1}\left[\frac{s+1}{s(s+2)}\right] = \frac{1}{2}L^{-1}\left[\frac{1}{s}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{2} + \frac{1}{2}e^{-2t}$$, using table of L.T.

Exercise-3 Find $L^{-1}\left\{\frac{a^2}{s(s+a)^2}\right\}$

Solution – Writing $\frac{a^2}{s(s+a)^2} \equiv \frac{A}{s} + \frac{B}{(s+a)} + \frac{C}{(s+a)^2}$

or $a^2 = (s+a)^2A + s(s+a)B + sC \qquad (1)$

Now comparing the coefficients of s^2, s and constant term on both the sides of (1), we get

$$A+B=0$$
$$2aA+aB+C=0$$
$$a^2A=a^2$$

Therefore, $A=1$, $B=-1$ and $C=-a$

So that

$$\frac{a^2}{s(s+a)^2} = \frac{1}{s} - \frac{1}{s+a} - \frac{a}{(s+a)^2}$$

Now $L^{-1}\left\{\frac{a^2}{s(s+a)^2}\right\} = 1 - e^{-at} - at\,e^{-at}$; using table of L.T.

<u>Heaviside Theorem</u> – Let $G(s)$ and $Q(s)$ be two polynomials in s where the degree of $G(s)$ is less than that of $Q(s)$, then the inverse Laplace transform of rational function $\frac{G(s)}{Q(s)}$ may be written as

$$L^{-1}\left[\frac{G(s)}{Q(s)}\right] = \sum_{i=1}^{k} \frac{G(\beta_i)}{Q'(\beta_i)}\, e^{\beta_i t}$$

where the prime represents derivatives w.r.t. s, β_i represents ith zero/root and k denotes the total number of distinct zeros/root of $Q(s)$.

<u>Proof</u> – Since $G(s)$ is a polynomial of degree less than that of $Q(s)$ and $Q(s)$ has k distinct zeros $\beta_1, \beta_2, \ldots, \beta_k$.

$$\therefore \frac{G(s)}{Q(s)} = \frac{G(s)}{(s-\beta_1)(s-\beta_2)\cdots(s-\beta_k)} \equiv \frac{A_1}{s-\beta_1} + \frac{A_2}{s-\beta_2} + \cdots + \frac{A_k}{s-\beta_k}$$

Multiplying both the sides by $(s-\beta_i)$ and putting $s=\beta_i$, we get

$$A_i = \lim_{s\to\beta_i} \frac{G(s)(s-\beta_i)}{Q(s)}$$

$$= G(\beta_i) \lim_{s\to\beta_i} \frac{(s-\beta_i)}{Q(s)}$$

$$= G(\beta_i) \lim_{s\to\beta_i} \frac{1}{Q'(s)}$$, by applying L'Hospital Rule.

$$= \frac{G(\beta_i)}{Q'(\beta_i)}$$

$$\therefore \frac{G(s)}{Q(s)} = \frac{G(\beta_1)}{Q'(\beta_1)} \frac{1}{s-\beta_1} + \frac{G(\beta_2)}{Q'(\beta_2)} \frac{1}{s-\beta_2} + \cdots + \frac{G(\beta_k)}{Q'(\beta_k)} \frac{1}{s-\beta_k}$$

Now $$L^{-1}\left[\frac{G(s)}{Q(s)}\right] = \frac{G(\beta_1)}{Q'(\beta_1)} e^{\beta_1 t} + \frac{G(\beta_2)}{Q'(\beta_2)} e^{\beta_2 t} + \cdots + \frac{G(\beta_k)}{Q'(\beta_k)} e^{\beta_k t}$$

$$= \sum_{i=1}^{k} \frac{G(\beta_i)}{Q'(\beta_i)} e^{\beta_i t}$$ Proved.

<u>Exercise-1</u> By using Heaviside theorem find

$$L^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}$$

<u>Solution</u> Here $G(s) = 3s+1$ and $Q(s) = (s-1)(s^2+1)$

Clearly, $Q(s)$ has three distinct roots $\beta_1=1$, $\beta_2=i$, $\beta_3=-i$

Also $Q'(s) = \frac{d}{ds}(s^3-s^2+s-1) = 3s^2-2s+1$

$\therefore$ By Heaviside theorem

$$L^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right] = \frac{G(\beta_1)}{Q'(\beta_1)} e^{\beta_1 t} + \frac{G(\beta_2)}{Q'(\beta_2)} e^{\beta_2 t} + \frac{G(\beta_3)}{Q'(\beta_3)} e^{\beta_3 t}$$

$$= \frac{4}{2} e^{t} + \frac{3i+1}{-2(i+1)} e^{it} + \frac{1-3i}{2(i-1)} e^{-it}$$

$$= 2e^{t} - \frac{3i+1}{2(i+1)} e^{it} + \frac{1-3i}{2(i-1)} e^{-it} \quad \underline{\text{Ans.}}$$

<u>Exercise-2</u> By using Heaviside theorem find

$$L^{-1}\left\{ \frac{2s^2-6s+5}{s^3-6s^2+11s-6} \right\}$$

<u>Solution-</u> Here $G(s) = 2s^2-6s+5$ and

$Q(s) = s^3-6s^2+11s-6 = (s-1)(s-2)(s-3)$

Clearly, $Q(s)$ has three distinct roots $\beta_1=1, \beta_2=2, \beta_3=3$ and $Q'(s) = 3s^2-12s+11$

Now by Heaviside theorem

$$L^{-1}\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6} \right] = \frac{1}{2} e^{t} - e^{2t} + \frac{5}{2} e^{3t} \quad \underline{\text{Ans.}}$$

<u>Exercise-3</u> By using Heaviside theorem find

(i) $L^{-1}\left[\frac{s^2-s+2}{s^3+4s^2+3s} \right]$ (ii) $L^{-1}\left[\frac{2s^2+5s-4}{s^3+s^2-2s} \right]$

(iii) $L^{-1}\left[\frac{3s^2+2s+1}{s^5-s} \right]$ (iv) $L^{-1}\left[\frac{s^2-3s}{s^3+s^2+s+1} \right]$

<u>Solution-</u> Try Yourself.

<u>Convolution Theorem-</u> Let $f(t)$ and $g(t)$ be two continuous functions. Also let $L[f(t)] = f^*(s)$ and $L[g(t)] = g^*(s)$. Then we know that

$$L[f(t) \circledcirc g(t)] = L\left[\int_0^t f(u)\, g(t-u)\, du \right] = f^*(s)\, g^*(s)$$

Therefore,

$$L^{-1}[f^*(s).g^*(s)] = \int_0^t f(u)\,g(t-u)\,du$$

This theorem is used to obtain inverse laplace transform of a function which can be expressed as the product of two or more functions. The following exercises will clear the application of this theorem.

Exercise-1 By using convolution theorem find $L^{-1}\left[\frac{1}{(s-1)(s+2)}\right]$

Solution- Let

$$f^*(s) = \frac{1}{s-1} \quad \text{and} \quad g^*(s) = \frac{1}{s+2}$$

From the table of L.T., we get

$$f(t) = e^{t} \quad \text{and} \quad g(t) = e^{-2t}$$

Now
$$L^{-1}\left[\frac{1}{(s-1)(s+2)}\right] = \int_0^t f(u)\,g(t-u)\,du$$
$$= \int_0^t e^{u} e^{-2(t-u)}\,du = e^{-2t}\int_0^t e^{3u}\,du$$
$$= \frac{1}{3}\left(e^{t} - e^{-2t}\right)$$

Exercise-2 By convolution theorem find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

Solution- Let

$$f^*(s) = \frac{s}{s^2+a^2} \quad \text{and} \quad g^*(s) = \frac{1}{s^2+a^2}$$

Obviously, $f(t) = \cos at$ and $g(t) = \frac{1}{a}\sin at$

Now $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \int_0^t f(u)\, g(t-u)\, du$

$$= \int_0^t \cos au \cdot \frac{1}{a} \sin a(t-u)\, du$$

$$= \frac{1}{a}\int_0^t \cos au\,(\sin at \cos au - \cos at \sin au)\, du$$

$$= \frac{1}{a} \sin at \int_0^t \cos^2 au\, du - \frac{1}{a} \cos at \int_0^t \sin au \cos au\, du$$

$$= \frac{1}{2a} \sin at \int_0^t (1+\cos 2au)\, du - \frac{1}{2a} \cos at \int_0^t \sin 2au\, du$$

$$= \frac{1}{2a} \sin at \left[t + \frac{\sin 2at}{2a}\right] - \frac{1}{2a} \cos at \left(1 - \frac{\cos 2at}{2a}\right)$$

$$= \frac{t \sin at}{2a} + \frac{\sin at \cdot 2 \sin at \cos at}{2a^2} -$$

<u>Exercise-4</u> By applying convolution theorem, find the value of

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

<u>Solution</u>— Let us consider

$$f(t) = \int_0^t x^{m-1}(t-x)^{n-1}\,dx$$

$$= \int_0^t f_1(x)\, f_2(t-x)\,dx\,, \quad \text{where } f_1(t) = t^{m-1} \text{ and } f_2(t) = t^{n-1}$$

$$= f_1(t) \circledcirc f_2(t)$$

Therefore,

$$f^*(s) = f_1^*(s)\cdot f_2^*(s)$$

$$= \frac{\Gamma(m)}{s^m}\cdot\frac{\Gamma(n)}{s^n} = \frac{\Gamma(m)\Gamma(n)}{s^{m+n}}.$$

and

$$f(t) = \int_0^t x^{m-1}(t-x)^{n-1}\,dx = \Gamma(m)\Gamma(n)\,L^{-1}\left[\frac{1}{s^{m+n}}\right]$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\, t^{m+n-1}.$$

Taking $t=1$, we get

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m,n).$$

The Solution of Simple Differential Equations by L.T. –

Example-1 Find the general solution of the differential equation $y''(t) + k^2 y(t) = 0$

Solution –

Taking Laplace Transform of both the sides

$$L[y''(t)] + k^2 L[y(t)] = 0$$

or $s^2 y^*(s) - s\,y(0) - y'(0) + k^2 y^*(s) = 0$

Let $y(0) = A$ and $y'(0) = B$

$\therefore\quad s^2 y^*(s) - As - B + k^2 y^*(s) = 0$

or $[s^2 + k^2]\, y^*(s) = As + B$

or $y^*(s) = A \dfrac{s}{s^2+k^2} + B \dfrac{1}{s^2+k^2}$

$$= A \frac{s}{s^2+k^2} + \frac{B}{k} \frac{k}{s^2+k^2}$$

Now using the table of L.T., we have

$$y(t) = A\,L^{-1}\left\{\frac{s}{s^2+k^2}\right\} + \frac{B}{k} L^{-1}\left\{\frac{k}{s^2+k^2}\right\}$$

$$= A \cos kt + \frac{B}{k} \sin kt$$

Example-2 Find the solution of the differential equation $y''(t) - y'(t) - 6y(t) = 2$ satisfying the initial conditions

$$y(0) = 1,\ y'(0) = 0$$

Solution – Taking Laplace transform of both the sides of differential equation we get

$$s^2 y^*(s) - s\,y(0) - y'(0) - s\,y^*(s) + y(0) - 6y^*(s) = \frac{2}{s}$$

or $s^2 y^*(s) - s - sy^*(s) + 1 - 6y^*(s) = \frac{2}{s}$

or $(s^2 - s - 6)\, y^*(s) = \frac{2}{s} + s - 1$

or $(s^2 - s - 6)\, y^*(s) = \frac{s^2 - s + 2}{s}$

or $y^*(s) = \frac{s^2 - s + 2}{s(s-3)(s+2)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s+2}$

Multiplying both the sides by s and putting $s=0$ we get $A = -1/3$

Multiplying both the sides by $s-3$ and putting $s=3$ we get $B = 8/15$

Similarly, multiplying both the sides by $s+2$ and putting $s=-2$, we get $C = 4/5$.

So $y^*(s) = -\frac{1}{3}\cdot\frac{1}{s} + \frac{8}{15}\cdot\frac{1}{s-3} + \frac{4}{5}\cdot\frac{1}{s+2}$

Taking inverse L.T., we get

$$y(t) = -\frac{1}{3} + \frac{8}{15} e^{3t} + \frac{4}{5} e^{-2t}$$ Ans.

Example-3 Find the functions $y(t)$ and $z(t)$ that satisfy the following system of differential eq^n.

$$y''(t) - z''(t) + z'(t) - y(t) = e^t - 2$$

$$2y''(t) - z''(t) - 2y'(t) + z(t) = -t$$

$$y(0) = y'(0) = z(0) = z'(0) = 0$$

Solution— Taking L.T. of both the differential equation and using the initial conditions, we get

$$s^2 y^*(s) - s^2 z^*(s) + s z^*(s) - y^*(s) = \frac{1}{s-1} - \frac{2}{s}$$

$$2s^2 y^*(s) - s^2 z^*(s) - 2s\, y^*(s) + z^*(s) = -\frac{1}{s^2}$$

or $(s^2 - 1)\, y^*(s) - (s^2 - s)\, z^*(s) = \frac{-s+2}{s(s-1)}$

$(2s^2 - 2s)\, y^*(s) - (s^2 - 1)\, z^*(s) = -\frac{1}{s^2}$

or $(s+1)\,y^*(s) - s\,z^*(s) = \dfrac{-s+2}{s(s-1)^2}$

$$2s\,y^*(s) - (s+1)\,z^*(s) = -\frac{1}{s^2(s-1)}$$

Simplifying the above algebric equation for $y^*(s)$ and $z^*(s)$, we get

$$y^*(s) = \frac{1}{s(s-1)^2} \quad \text{and} \quad z^*(s) = \frac{2s-1}{s^2(s-1)^2}$$

Taking the inverse L.T. of $y^*(s)$ and $z^*(s)$, we find

$$y(t) = 1 - e^t + te^t$$

and $z(t) = -t + te^t$ **Ans.**

Problems—

Find the solution of the following differential equations—

(i) $y''(t) - k^2 y(t) = 0 \quad (k \neq 0)$

(ii) $y''(t) - 2k\,y'(t) + k^2 y(t) = 0$

(iii) $y''(t) + k^2 y(t) = a$

(iv) $y''(t) + 2y'(t) + 2y(t) = 0$, $y(0) = 0$, $y'(0) = 1$

(v) $y''(t) + y'(t) = t^2 + 2t$, $y(0) = 4$, $y'(0) = -2$

(vi) $x'(t) + y'(t) + x(t) + y(t) = 1$

$y'(t) - 2x(t) - y(t) = 0$

$x(0) = 0$, $y(0) = 1$