Laplace-Stieltjes Transform - The Laplace-Stieltjes transform (LST) of a continuous function $f(t)$ is defined as

$$\tilde{f}(s) \equiv \int_0^\infty e^{-st}\, df(t)$$

We can relate $\tilde{f}(s)$ to $f^*(s)$ [when $f^*(s)$ exists] by integrating by parts as follows -

$$\tilde{f}(s) = \int_0^\infty e^{-st}\, df(t) = e^{-st} f(t)\Big|_0^\infty + \int_0^\infty s e^{-st} f(t)\, dt$$

Assuming $f(t)$ is such that as $t \to \infty$, it does not go to ∞ as fast as e^{-t} approaches to zero, i.e. $\lim_{t\to\infty} e^{-st} f(t) = 0$

So, we have

$$\tilde{f}(s) = s f^*(s) - f(0) = L[f'(t)]$$

If we consider the LST of a continuous c.d.f., $F(\cdot)$ where the random variable is non-negative, then $F(\infty)=1$ and $F(0)=0$, so that

$$\tilde{F}(s) = sF^*(s)$$

and

$$\tilde{F}(s) = \int_0^\infty e^{-st}\, dF(t)$$

$$= \int_0^\infty e^{-st} f(t)\, dt = f^*(s)$$

i.e. LST of a Cdf = LT of the corresponding p.d.f.

Laplace-Stieltjes Transform of Stieltjes Convolution -

The integral

$$\int_0^t dF(u)\, G(t-u) = F(t) \circledS G(t)$$

is called the Stieltjes convolution of two continuous functions $F(t)$ and $G(t)$.

The LST of Stieltjes convolution is

$$LS[F(t) \circledS G(t)] = \int_0^\infty e^{-st}\, d\int_0^t dF(u)G(t-u)$$

$$= e^{-st}\int_0^t dF(u)\,G(t-u)\Big|_0^\infty + s\int_0^\infty e^{-st}dt\,(\int_0^t dF(u)G(t-u)]$$

$$= 0 + s\int_0^\infty e^{-st}dt \int_0^t F'(u)G(t-u)du$$

$$= s\int_0^\infty e^{-st}dt\,[F'(t) \copyright G(t)]$$

$$= s\,L.T.[F'(t) \copyright G(t)]$$

$$= s\,L.T.[F'(t)].\,L.T.[G(t)]$$

$$= L.S.T.[F(t)].\,s\,L.T.[G(t)]$$

$$= \tilde{F}(s).\tilde{G}(s) \quad , \text{ if } G(0)=0$$

Mean and Variance in terms of L.T. —

Let $f(t)$ be a p.d.f. of a non-negative continuous r.v. T. Then

$$\frac{d}{ds} f^*(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt$$

$$= -\int_0^\infty t\,e^{-st} f(t)dt$$

$$\frac{d^2}{ds^2} f^*(s) = (-1)^2\int_0^\infty t^2 e^{-st} f(t)dt$$

In general

$$\frac{d^n}{ds^n} f^*(s) = (-1)^n\int_0^\infty t^n e^{-st} f(t)dt$$

If we put $s=0$, we get

$$\frac{d^n}{ds^n} f^*(s)\Big|_{s=0} = (-1)^n\int_0^\infty t^n f(t)dt = (-1)^n E(T^n)$$

or $\quad E(T^n) = (-1)^n \dfrac{d^n}{ds^n} f^*(s)\Big|_{s=0}$

In particular

$$E(T) = -\frac{d}{ds} f^*(s)\Big|_{s=0} \quad , \quad E(T^2) = \frac{d^2}{ds^2} f^*(s)\Big|_{s=0}$$

so that $V(T) = E(T^2) - [E(T)]^2$

Example-1 Let X follows negative exponential distribution with parameter λ. Its density function is

$$f(x) = \lambda e^{-\lambda x}, \lambda > 0 \ ; \ 0 \leq x < \infty$$
$$= 0 \qquad ; \ x < 0$$

The L.T. of $f(x)$ is

$$f^*(s) = \int_0^\infty e^{-sx} f(x)\, dx$$
$$= \lambda \int_0^\infty e^{-sx} e^{-\lambda x} dx = \frac{\lambda}{s+\lambda}$$

Now, we have

$$\frac{d}{ds} f^*(s) = -\frac{\lambda}{(s+\lambda)^2} \quad , \quad \frac{d^2}{ds^2} f^*(s) = \frac{2\lambda}{(s+\lambda)^3}$$

Therefore,

$$E(X) = -\frac{d}{ds} f^*(s)\Big|_{s=0} = \frac{1}{\lambda} \text{ and } E(X^2) = \frac{d^2}{ds^2} f^*(s)\Big|_{s=0} = \frac{2}{\lambda^2}$$

so that $V(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

Example-2 Let X follows negative exponential distribution with parameter λ, μ. Its density function is

$$f(x) = \lambda e^{-\lambda(x-\mu)}, \lambda, \mu > 0 \ ; \mu \leq x < \infty$$
$$= 0 \qquad ; \ x < \mu$$

The L.T. of $f(x)$ is

$$f^*(s) = \int_0^\infty e^{-sx} f(x)\,dx$$

$$= \int_0^\mu e^{-sx} f(x)\,dx + \int_\mu^\infty e^{-sx} f(x)\,dx$$

$$= \lambda \int_\mu^\infty e^{-sx} e^{-\lambda(x-\mu)}\,dx$$

$$= \lambda e^{\lambda\mu} \int_\mu^\infty e^{-(s+\lambda)x}\,dx$$

$$= \lambda e^{\lambda\mu}\left(\frac{e^{-(s+\lambda)\mu}}{s+\lambda}\right) = \frac{\lambda}{s+\lambda} e^{-s\mu}$$

Now,
$$\frac{d}{ds} f^*(s) = \frac{\lambda}{s+\lambda}(-\mu)e^{-s\mu} + e^{-s\mu}(-1)\frac{\lambda}{(s+\lambda)^2}$$

and
$$\frac{d^2}{ds^2} f^*(s) = (-\mu)\left[\frac{\lambda}{s+\lambda}(-\mu)e^{-s\mu} + e^{-s\mu}(-1)\frac{\lambda}{(s+\lambda)^2}\right]$$
$$-\lambda\left[e^{-s\mu}(-2)\frac{1}{(s+\lambda)^3} + \frac{1}{(s+\lambda)^2}(-\mu)e^{-s\mu}\right]$$

So that

$$E(X) = -\frac{d}{ds} f^*(s)\Big|_{s=0} = \frac{1}{\lambda} + \mu$$

and
$$E(X^2) = \frac{d}{ds^2} f^*(s)\Big|_{s=0} = \mu^2 + \frac{\mu}{\lambda} + \frac{2}{\lambda^2} + \frac{\mu}{\lambda}$$
$$= \mu^2 + \frac{2}{\lambda^2} + \frac{2\mu}{\lambda}$$

Therefore, $V(X) = \mu^2 + \frac{2}{\lambda^2} + \frac{2\mu}{\lambda} - \left(\frac{1}{\lambda} + \mu\right)^2 = \frac{1}{\lambda^2}$ (indep. of μ)

Example-3 Let X follows two parameter gamma distribution with parameters λ, k ($\lambda > 0$ is the scale parameter and $k > 0$ is the shape parameter).

The density function of the r.v. X is

$$f(x) = \frac{\lambda^k e^{-\lambda x} x^{k-1}}{\Gamma(k)}, \quad x \geq 0$$
$$= 0, \quad x < 0$$

The L.T. $f^*(s)$ is given by

$$f^*(s) = \int_0^\infty e^{-sx} f(x)\, dx$$
$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty e^{-sx} e^{-\lambda x} x^{k-1}\, dx$$
$$= \frac{\lambda^k}{\Gamma(k)} \frac{\Gamma(k)}{(s+\lambda)^k}, \text{(by gamma integral)}$$
$$= \left(\frac{\lambda}{s+\lambda}\right)^k$$

Now, we have

$$\frac{d}{ds} f^*(s) = \frac{-k\lambda^k}{(s+\lambda)^{k+1}} \quad \text{and} \quad \frac{d^2}{ds^2} f^*(s) = \frac{k(k+1)\lambda^k}{(s+\lambda)^{k+2}}$$

Therefore,

$$E(X) = -\frac{d}{ds} f^*(s)\Big|_{s=0} = \frac{k}{\lambda}, \quad E(X^2) = \frac{d^2}{ds^2} f^*(s)\Big|_{s=0} = \frac{k(k+1)}{\lambda^2}$$

so that, $V(X) = \dfrac{k}{\lambda^2}$

If we replace λ by λk in the above gamma density, we get the density of Erlang distribution. i.e.

$$f(x) = \frac{(\lambda k)^k e^{-\lambda k x} x^{k-1}}{\Gamma(k)}, \quad x \geq 0$$

In this case $f^*(s) = \left(\dfrac{\lambda k}{s+\lambda k}\right)^k$

and $E(X) = \dfrac{1}{\lambda}$, $V(X) = \dfrac{1}{k\lambda}$.

Example-4 Let X be a random variable whose whole mass is concentrated at one single point say 'a' i.e. the r.v. X follows one point distribution as follows -

$$P(X=a) = 1 \text{ and } P(X \neq a) = 0$$

The L.T. of this distribution is given by

$$f^*(s) = E[e^{-sX}]$$
$$= e^{-sa} P(X=a) = e^{-sa}$$

Now

$$\frac{d}{ds} f^*(s) = (-a) e^{-sa} \quad \text{and} \quad \frac{d^2}{ds^2} f^*(s) = a^2 e^{-sa}$$

so that,

$$E(X) = -\frac{d}{ds} f^*(s) \Big|_{s=0} = a$$

and $$E(X^2) = \frac{d^2}{ds^2} f^*(s) \Big|_{s=0} = a^2$$

Therefore, $V(X) = a^2 - a^2 = 0$

Example-5 Let X follows Poisson distribution with parameter λ i.e.

$$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!} \; ; \; x = 0, 1, 2, \ldots$$

The L.T. of this probability function is given by

$$f^*(s) = E[e^{-sX}]$$
$$= \sum_{x=0}^{\infty} e^{-sx} P(X=x)$$

$$= \sum_x e^{-sx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{-s})^x}{x!} = e^{-\lambda} e^{\lambda e^{-s}}$$

$$= \exp[\lambda(e^{-s}-1)]$$

Now

$$E(X) = -\frac{d}{ds} f^*(s)\Big|_{s=0} = -1\left[e^{\lambda(e^{-s}-1)} \lambda e^{-s}(-1)\right]\Big|_{s=0}$$

$$= \lambda$$

$$E(X^2) = \frac{d^2}{ds^2} f^*(s)\Big|_{s=0} = (-\lambda)\left[e^{-s} e^{\lambda(e^{-s}-1)} \lambda e^{-s}(-1) + e^{\lambda(e^{-s}-1)} e^{-s}(-1)\right]\Big|_{s=0}$$

$$= (-\lambda)[-\lambda - 1] = \lambda^2 + \lambda$$

Therefore,

$$V(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

<u>Theorem</u> — If $L[f(t)] = f^*(s)$, then

$$L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} f^*(u)\,du$$

provided $\lim_{t \to 0} \left\{\frac{f(t)}{t}\right\}$ exists.

<u>Proof</u> — Let $\frac{f(t)}{t} = g(t)$

$\therefore \quad f(t) = t\,g(t)$

Now $L[f(t)] = L[t\,g(t)] = -\frac{d}{ds} L[g(t)]$

or $\quad f^*(s) = -\frac{d}{ds} g^*(s) \qquad (1)$

Integrating both sides of (1) w.r.t. s from s to ∞, we have

$$\int_s^\infty f^*(u)\,du = -[g^*(u)]_s^\infty$$
$$= g^*(s) - g^*(\infty)$$

or $\quad g^*(s) = \int_s^\infty f^*(u)\,du \qquad \because g^*(\infty) = 0$

Proved

Ex. 1 Find L.T of the following functions –

(i) $f(t) = t^n \quad (n > -1)$

(ii) $f(t) = e^{at}$

(iii) $f(t) = t^n e^{at} \quad (n > -1)$

(iv) $f(t) = \sin kt$

Solⁿ (1) The L.T of $f(t) = t^n$ is given by

$$L[t^n] = \int_0^\infty e^{-st}\, t^n\, dt$$
$$= \frac{\Gamma(n+1)}{s^{n+1}}$$

(ii) The L.T of $f(t) = e^{at}$ is

$$L[e^{at}] = \int_0^\infty e^{-st}\, e^{at}\, dt$$
$$= \int_0^\infty e^{-(s-a)t}\, t^{1-1}\, dt$$
$$= \frac{1}{(s-a)}$$

(iii) The L.T. of $f(t) = t^n e^{at}$ is

$$L[t^n e^{at}] = \int_0^\infty e^{-st}\, t^n e^{at}$$

Table of Laplace Transforms of some functions

S.N.	$f(t)$	$f^*(s)$	S.N.	$f(t)$	$f^*(s)$
1	1	$1/s$	11	$\cos nt$	$s/(s^2+n^2)$
2	$a\,f(t)$	$a\,f^*(s)$	12	$\sinh nt$	$n/(s^2-n^2)$
3	$f_1(t)+f_2(t)$	$f_1^*(s)+f_2^*(s)$	13	$\cosh nt$	$s/(s^2-n^2)$
4.	$f(t-a),\ a>0$	$e^{-sa}\,f^*(a)$	14	$e^{-at}\sin nt$	$\frac{n}{(s+a)^2+n^2},\ s>-a$
5.	$t\,f(t)$	$-\frac{d}{ds}\,f^*(s)$	15	$e^{-at}\cos nt$	$\frac{s+a}{(s+a)^2+n^2},\ s>-a$
6.	$e^{-at}f(t)$	$f^*(s+a)$	16	$\cancel{t^n e^{-at}\ (n>-1)}$	$\cancel{\Gamma(n+1)/(s+a)^{n+1}}$
7	e^{-at}	$1/(s+a)$	17	$e^{-at}\,t^n/n!$	$1/(s+a)^{n+1},\ s>-a$
8.	$t^n\ (n>-1)$	$\Gamma(n+1)/s^{n+1}$	18	$e^{at}-e^{bt}\ (a>b)$	$\frac{a-b}{(s-a)(s-b)},\ s>a$
9.	$t^n e^{-at}$	$\Gamma(n+1)/(s+a)^{n+1}$	19	$f(t/a)$	$a\,f^*(as)$
10	$\sin nt$	$n/(s^2+n^2)$	20	$f'(t)$	$s\,f^*(s)-f(0)$