Example 7.10: ... output $y(n)$ of a linear time invariant discrete time system specified by the equation.

$$y(n)-\frac{3}{2}y(n-1)+\frac{1}{2}y(n-2)=2x(n)+\frac{3}{2}x(n-1)$$

if the initial condition are $y(-1)=0, y(-2)=1$ and the input $x(n)=\left(\frac{1}{4}\right)^n u(n)$.

Solution : Given

$$y(n)-\frac{3}{2}y(n-1)+\frac{1}{2}y(n-2)=2x(n)+\frac{3}{2}x(n-1)$$

Taking z-transform on both sides

$$y(z)-\frac{3}{2}[z^{-1}Y(z)+y(-1)]+\frac{1}{2}[z^{-2}Y(z)+z^{-1}y(-1)+y(-2)]$$

$$=2X(z)+\frac{3}{2}[z^{-1}X(z)+x(-1)] \quad ...(1)$$

We have

$$y(-1)=0;\ y(-2)=1$$

and $$x(-1)=0$$

Substituting above values in Equation (1), we get

$$Y(z)-\frac{3}{2}z^{-1}Y(z)+\frac{1}{2}\left[z^{-2}\,Y(z)+1\right]=X(z)\left[2+\frac{3}{2}z^{-1}\right]$$

For input $$x(n)=\left(\frac{1}{4}\right)^n u(n)$$

$$X(z)=\frac{1}{1-\frac{1}{4}z^{-1}}$$

$$\Rightarrow \quad Y(z)\left[1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}\right]=-\frac{1}{2}+\frac{1}{1-\frac{1}{4}z^{-1}}\left(2+\frac{3}{2}z^{-1}\right)$$

$$Y(z)=\frac{-1}{2\left(1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-1}\right)}+\frac{\left(2+\frac{3}{2}z^{-1}\right)}{\left(1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}\right)}$$

$$=\frac{-z^2}{2\left(z^2-\frac{3}{2}z+\frac{1}{2}\right)}+\frac{z^2\left(2z+\frac{3}{2}\right)}{\left(z-\frac{1}{4}\right)\left(z^2-\frac{3}{2}z+\frac{1}{2}\right)}$$

$$=\frac{-z^2}{2(z-1)\left(z-\frac{1}{2}\right)}+\frac{z^2\left(2z+\frac{3}{2}\right)}{\left(z-\frac{1}{4}\right)(z-1)\left(z-\frac{1}{2}\right)}$$

$$=Y_1(z)+Y_2(z)$$

$$Y_1(z) = \frac{-z^2}{2(z-1)\left(z-\frac{1}{2}\right)}$$

$$\frac{Y_1(z)}{z} = \frac{-z}{2(z-1)\left(z-\frac{1}{2}\right)}$$

$$= \frac{A}{z-1} + \frac{B}{z-\frac{1}{2}}$$

$$= \frac{-1}{z-1} + \frac{1}{2\left(z-\frac{1}{2}\right)}$$

$$Y_1(z) = \frac{-z}{z-1} + \frac{z}{2\left(z-\frac{1}{2}\right)}$$

$$y_1(z) = -u(n) + 0.5\left(\frac{1}{2}\right)^n u(n)$$

$$\frac{Y_2(z)}{z} = \frac{z^2\left(2z+\frac{3}{2}\right)}{\left(z-\frac{1}{4}\right)(z-1)\left(z-\frac{1}{2}\right)}$$

$$= \frac{A_1}{z-\frac{1}{4}} + \frac{B_1}{z-1} + \frac{C_1}{z-\frac{1}{2}}$$

$$= \frac{8}{3\left(z-\frac{1}{4}\right)} + \frac{28}{3(z-1)} - \frac{10}{z-\frac{1}{2}}$$

$$= \frac{8}{3}\frac{z}{z-\frac{1}{4}} + \frac{28}{3}\frac{z}{z-1} - 10\frac{z}{z-\frac{1}{2}}$$

$$y_2(n) = \frac{8}{3}\left(\frac{1}{4}\right)^n u(n) + \frac{28}{3}u(n) - 10\left(\frac{1}{2}\right)^n u(n)$$

$$y(n) = \frac{25}{3}u(n) + \frac{8}{3}\left(\frac{1}{4}\right)^n u(n) - \frac{19}{2}\left(\frac{1}{2}\right)^n u(n)$$

$$A = \cancel{(z-1)} \cdot \left.\frac{-z}{z\cancel{(z-1)}\left(z-\frac{1}{2}\right)}\right|_{z=1}$$

$$= \frac{1}{2\left(1-\frac{1}{2}\right)} = -1$$

$$B = \cancel{\left(z-\frac{1}{2}\right)} \left.\frac{-z}{z(z-1)\cancel{\left(z-\frac{1}{2}\right)}}\right|_{z=\frac{1}{2}}$$

$$= \frac{\frac{1}{2}}{2\left(\frac{1}{2}-1\right)} = \frac{\frac{1}{2}}{2\left(-\frac{1}{2}\right)} = -\frac{1}{2}$$

$$A_1 = \cancel{\left(z-\frac{1}{4}\right)} \left.\frac{z\left(2z+\frac{3}{2}\right)}{\cancel{\left(z-\frac{1}{4}\right)}(z-1)}\right|_{z=\frac{1}{4}}$$

$$= \frac{\frac{1}{4}\left(\frac{1}{2}+\frac{3}{2}\right)}{\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-\frac{1}{2}\right)} = \frac{\frac{1}{4}(2)}{\left(\frac{-3}{4}\right)\left(-\frac{1}{4}\right)}$$

$$= \frac{1}{2}\cdot\frac{16}{3} = \frac{8}{3}$$

$$B_1 = \cancel{(z-1)} \frac{z\left(2z+\frac{3}{2}\right)}{\cancel{(z-1)}\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}$$

$$= \frac{1\left(2+\frac{3}{2}\right)}{\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)} = \frac{\frac{7}{2}}{\frac{3}{8}} = \frac{7}{2}\cdot\frac{8}{3} = \frac{28}{3}$$

$$C_1 = \cancel{\left(z-\frac{1}{2}\right)} \frac{z\left(2z+\frac{3}{2}\right)}{(z-1)\left(z-\frac{1}{4}\right)\cancel{\left(z-\frac{1}{2}\right)}}$$

$$= \frac{\frac{1}{2}\left(1+\frac{3}{2}\right)}{\left(-\frac{1}{2}\right)\left(\frac{1}{4}\right)} = \frac{5}{4}\left(\frac{-8}{1}\right) = -10$$

Example. 7.71. Using z-transform determine the response of the linear time-invariant system with difference equation.

$y(n) - 2r\cos\theta\, y(n-1) + r^2 y(n-2) = x(n)$

to an excitation $x(n) = a^n u(n)$

Solution : Given $y(n) - 2r\cos\theta\, y(n-1) + r^2 y(n-2) = x(n)$

Taking z-transform and both side and applying initial conditions to zero yields

$Y(z) - 2r\cos\theta z^{-1} Y(z) + r^2 z^{-2} Y(z) = X(z)$

$$\Rightarrow \quad \frac{Y(z)}{X(z)} = \frac{1}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

For $\quad x(n) = a^n u(n)$

$$X(z) = \frac{1}{1 - az^{-1}}$$

i.e.,
$$Y(z) = \frac{1}{(1-az^{-1})(1 - 2r\cos\theta z^{-1} + r^2 z^{-2})}$$

$$= \frac{1}{(1-az^{-1})(1 - re^{j\theta} z^{-1})(1 - re^{-j\theta z^{-1}})}$$

$$= \frac{z^3}{(z-a)(z-re^{j\theta})(z-re^{-j\theta})}$$

$$\frac{Y(z)}{z} = \frac{z^2}{(z-a)(z-re^{j\theta})(z-re^{-j\theta})} = \frac{A}{z-a} + \frac{B}{z-re^{j\theta}} + \frac{C}{z-re^{-j\theta}}$$

$$Y(z) = \frac{a^2}{a^2 - 2ar\cos\theta + r^2}\cdot\left(\frac{z}{z-a}\right) + \frac{r^2 e^{j2\theta}}{(re^{j\theta}-a)(re^{j\theta}-re^{-j\theta})}\frac{z}{z-re^{j\theta}}$$

$$+ \frac{r^2 e^{-j2\theta}}{(re^{-j\theta}-a)(re^{-j\theta}-re^{j\theta})}\frac{z}{z-re^{-j\theta}}$$

Taking inverse z-transform on both sides and simplifying, we get

$$y(n) = \frac{a^2}{a^2 - 2ar\cos\theta + r^2} a^n u(n) + \frac{r^{n+1}}{\sin\theta}\left[\frac{r\sin(n+1)\theta - a\sin(n+2)\theta}{a^2 - 2ar\cos\theta + r^2}\right] u(n)$$

Example 7.72. Find the response of

$y(n) + y(n-1) - 2y(n-2) = u(n-1) + 2u(n-2)$

due to $y(-1) = 0.5;\ y(-2) = 0.25$

Solution : Given

$y(n) + y(n-1) - 2y(n-2) = u(n-1) + 2u(n-2)$

Take z-transform on both sides

$Y(z) + [z^{-1} Y(z) + y(-1)] - 2[z^{-2} Y(z) + z^{-1} y(-1) + y(-2)]$

$$= \frac{z^{-1}}{1-z^{-1}} + \frac{2z^{-2}}{1-z^{-1}}$$

$\Rightarrow Y(z) + z^{-1} Y(2) + 0.5 - 2[z^{-2} Y(z) + 0.5z^{-1} + 0.25]$

$$= \frac{z^{-1}}{1-z^{-1}} + \frac{2z^{-2}}{1-z^{-1}}$$

$$Y(z)[1 + z^{-1} - 2z^{-2}] = z^{-1} + \frac{z^{-1}}{1-z^{-1}} + \frac{2z^{-2}}{1-z^{-1}}$$

$$Z[u(n)] = \frac{1}{1-z^{-1}}$$

$$Z(u(n-1)) = \frac{z^{-1}}{1-z^{-1}}$$

$$Y(z) = \frac{z^{-1}}{1 - z^{-1} - 2z^{-2}} + \frac{z^{-1} + 2z^{-2}}{(1 - z^{-1})(1 + z^{-1} - 2z^{-2})}$$

$$= \frac{z}{z^2 + z - 2} + \frac{z(z+2)}{(z-1)(z^2 + z - 2)}$$

$$Y(z) = Y_1(z) + Y_2(z)$$

Taking inverse z-transform, we get

$$y(n) = y_1(n) + y_2(n)$$

$$Y_1(z) = \frac{z}{(z+2)(z-1)}$$

$$\frac{Y_1(z)}{z} = \frac{A}{z+2} + \frac{B}{z-1}$$

$$= \frac{-1}{3(z+2)} + \frac{1}{2(z-1)}$$

$$Y_1(z) = \frac{-1}{3}\frac{z}{z+2} + \frac{1}{3}\frac{z}{z-1}$$

$$y_1(n) = \frac{-1}{3}(-2)^n u(n) + \frac{1}{3}u(n)$$

$$Y_2(z) = \frac{z(z+2)}{(z-1)^2(z+2)}$$

$$y^2(n) = nu(n)$$

$$\Rightarrow \quad y(n) = \frac{1}{3}u(n) + nu(n) - \frac{1}{3}(-2)^n u(n)$$

$$A = (z+2)\left.\frac{1}{(z+2)(z-1)}\right|_{z=-2}$$

$$= \frac{-1}{3}$$

$$B = (z-1)\left.\frac{1}{(z+2)(z-1)}\right|_{z=1}$$

$$= \frac{1}{3}$$