$$= \ldots + x_1(-2)\, z^2 + x_1(-1)\, z^1 + x_1(0)\, z^0 + x_1(1)\, z^{-1} + x_1(2)\, z^{-2} + x_1(3)\, z^{-1} + x_1(4)\, z^{-4} + \ldots$$

or $$X_1(z) = 4 - 2z^{-1} + z^{-2} \quad \ldots(i)$$

Also, $$X_2(z) = Z[x_2(n)] = \sum_{n=-\infty}^{\infty} x_2(n)\, z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} \quad \ldots(ii)$$

From convolution property of z-transform, we know that

$$X(z) = X_1(z)\, X_2(z) \quad \ldots(iii)$$

Substituting, equations (i) and (ii) in equation (iii), we obtain

or $$X(z) = (4 - 2z^{-1} + z^{-2})\,(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5})$$

$$X(z) = (4 + 2z^{-1} + 3z^{-2} + .3z^{-3} + 3z^{-4} + 3z^{-5} + z^{-6} + z^{-7} \quad \ldots(iv)$$

Now, taking the inverse z-transform of equation (iv), we have

$$x(n) = Z^{-1}\,[X(z)] = \{\underset{\uparrow}{4}, 2, 3, 3, 3, 3, -1, 1\} \quad \textbf{Ans.}$$

Example 7.52. Find z-transform of the following discrete-time signal

$$\boldsymbol{x(n) = 2^n u(n-2)}$$

Solution : We know that

$$Z\,[u(n)] = \frac{1}{1-z^{-1}}$$

Therefore, the z-transform of u(n − 2) can be found as under :

$$Z\,[u(n-2)] = z^{-2}\, Z\,[u(n)]$$

$$= z^{-2}\left(\frac{1}{1-z^{-1}}\right) \quad \ldots(i)$$

Therefore, $Z\,[2^n\, u(n-2)] = Z[u(n-2)]\big|_{z^{-1} \to 2z^{-1}}$

$$= \left.\frac{z^{-2}}{1-z^{-1}}\right|_{z^{-1} \to 2z^{-1}} = \frac{(2z^{-1})^2}{1-2z^{-1}} = \frac{4z^{-2}}{1-2z^{-1}} \quad \textbf{Ans.}$$

Example 7.53. Obtain the cross-correlation sequence $r_{x_1x_2}(l)$ of the following given sequences

$$\boldsymbol{x_1(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}}$$

$$\boldsymbol{x_2(n) = \{\underset{\uparrow}{4}, 3, 2, 1\}}$$

Solution : We know that the Cross-correlation sequence can be obtained by using the correlation property of z-transform, *i.e.*

$$R_{x_1x_2}(z) = X_1(z)\, X_2(z^{-1}) \quad \ldots(i)$$

Now, taking z-transforms of sequence $x_1(n)$ and $x_2(n)$, we obtain

$$X_1(z) = Z\,[x_1(n)] = 1\ 2z^{-1} + 3z^{-2}\ 4\, z^{-3} \quad \ldots(ii)$$

$$X_2(z) = Z\,[x_2(n)] = 4 + 3z^{-1} + 2z^{-2} + z^{-3} \quad \ldots(iii)$$

Substituting $z = z^{-1}$ in equation (iii), we get

$$X_2(z^{-1}) = 4 + 3z + 2z^2 + z^3 \quad \ldots(iv)$$

Substituting equations (*ii*) and (*iv*) in equation (*i*), we get

$$R_{x_1x_2}(z) = X_1(z)\,X_2(z^{-1}) = (1 + 2z^{-1} + 3z^{-2} + 4z^{-3})\,(4 + 3z + 2z^2 + z^3)$$

or $$R_{x_1x_2}(z) = (z^3 + 4z^2 + 10z + 20 + 25z^{-1} + 24z^{-2} + 16z^{-3}) \quad ...(v)$$

Taking the inverse z-transform of equation (*v*), we get

$$r_{x_1x_2}(l) = Z^{-1}[R_{x_1x_2}(z)] = \{1, 4, 10, \underset{\uparrow}{20}, 25, 24, 16\}$$ **Ans.**

Example 7.54. Find the initial and final values of the corresponding sequence, $x(n)$. Given

$$X(z) = 2 + 3z^{-1} + 4z^{-2}.$$

Solution : We know that, the initial value of the sequence $x(n)$ is given by

$$x(0) = \lim_{|z|\to\infty}[2 + 3z^{-1} + 4z^{-2}] = 2 + \frac{3}{\infty} + \frac{4}{\infty} = 2$$

Further, final value of the sequence $x(n)$ is given by

$$x(\infty) = \lim_{|z|\to 1}[(1 - z^{-1})\,(2 + 3z^{-1} + 4z^{-2})]$$

$$= \lim_{|z|\to 1}[2 + z^{-1} + z^{-2} - 4z^{-3}]$$

or $$x(\infty) = 2 + (1)^{-1} + (1)^{-2} - 4(1)^{-3} = 2 + 1 + 1 - 4 = 0$$

Initial and final values of the signals are 2 and 0 respectively. **Ans.**

Example 7.55. Find the causal discrete-time signal $x(n)$ for $X(z)$ which is given by

$$X(z) = \frac{1 + 2z^{-1}}{1 - 2z^{-1} + 4z^{-2}}$$

Solution : Given that :

$$X(z) = \frac{1 + 2z^{-1}}{1 - 2z^{-1} + 4z^{-2}} \quad ...(i)$$

Equation (*i*) can be written as under :

$$X(z) = \frac{z^2 + 2z}{z^2 - 2z + 4} \quad ...(ii)$$

We know that

$$Z[A^n \cos\omega_0 n] = \frac{z(z - A\cos\omega_0)}{z^2 - 2Az\cos\omega_0 + A^2}$$

and $$Z[A^n \sin\omega_0 n] = \frac{Az\sin\omega_0}{z^2 - 2Az\cos\omega_0 + A^2}$$

Now, the denominator of $X(z)$ gives

$$z^2 - 2z + 4 = z^2 - 2Az\cos\omega_0 + A^2$$

or $$A\cos\omega_0 = 1 \text{ and } A = 2$$

$$2\cos\omega_0 = 1$$

or $$\cos\omega_0 = \frac{1}{2} = \cos 60°$$

or $$\omega_0 = 60° = \frac{\pi}{3}$$

$$\sin\omega_0 = \sin 60° = \frac{\sqrt{3}}{2}$$

Also, the numerator of $X(z)$ gives

$$z^2 + 2z = z^2 - Az\cos\omega_0 + \alpha\,(Az\sin\omega_o)$$

or
$$z^2 + 2z = z^2 - 2z\left(\frac{1}{2}\right) + \alpha\left(2z\frac{\sqrt{3}}{2}\right)$$

or
$$\alpha = \sqrt{3}$$

Therefore, we have

$$X(z) = \frac{\left[z^2 - 2z\cos\frac{\pi}{3}\right] + \sqrt{3}\left[2z\sin\frac{\pi}{3}\right]}{z^2 - 2z\left(2\cos\frac{\pi}{3}\right) + 4}$$

Thus, we get $$x(n) = 2^n\left[\cos\frac{n\pi}{3} + \sqrt{3}\sin\frac{n\pi}{3}\right]u(n) \quad \textbf{Ans.}$$

Example 7.56. Impulse response of a discrete-time LTI system is expressed as under :

$$h(n) = \{\underset{\uparrow}{1}, 2, 3\}$$

Find the input sequence $x(n)$ for output response which is given by

$$y(n) = \{\underset{\uparrow}{1}, 1, 2, -1, 3\}$$

Solution : Taking the z-transform of $h(n)$ and $y(n)$, we obtain

$$H(z) = Z\,[h(n)] = \sum_{n=0}^{2} h(n)\,z^{-n}$$

$$= h(0)\,z^{-0} + h(1)\,z^{-1} + h(2)\,z^{-2}$$

$$= 1 + 2z^{-1} + 3z^{-2} \quad \text{...(i)}$$

and
$$Y(z) = Z\,[y(n)] = \sum_{n=0}^{4} y(n)\,z^{-n}$$

$$= y(0)\,z^{-0} + y(1)\,z^{-1} + y(2)\,z^{-2} + y(3)\,z^{-3} + y(4)\,z^{-4}$$

or
$$Y(z) = 1 + z^{-1} + 2z^{-2} + z^{-3} + 3z^{-4} \quad \text{...(ii)}$$

Further, we know that

$$Y(z) = X(z)\,H(z)$$

or
$$X(z) = \frac{Y(z)}{H(z)} \quad \text{...(iii)}$$

Substituting equations (*i*) and (*ii*) in equation (*iii*), we obtain

$$X(z) = \frac{1 + z^{-1} + 2z^{-2} - z^{-3} + 3z^{-4}}{1 + 2z^{-1} + 3z^{-2}} \quad \text{...(iv)}$$

$$\begin{array}{r|l} 1+2z^{-1}+3z^{-2} & 1+z^{-1}+2z^{-2}-z^{-3}+3z^{-4} \;(1-z^{-1}+z^{-2} \\ & 1+2z^{-1}+3z^{-2} \\ & -\quad -\quad - \\ \hline & -z^{-1}-z^{-2}-z^{-3} \\ & -z^{-1}-2z^{-2}-3z^{-3} \\ & +\quad +\quad + \\ \hline & z^{-2}+2z^{-3}+3z^{-4} \\ & z^{-2}+2z^{-3}+3z^{-4} \\ & -\quad -\quad - \\ \hline & 0 \end{array}$$

Therefore, we have $X(z) = 1 - z^{-1} + z^{-2}$ ...(v)

Taking the inverse z-transform of equation (v), we obtain

$$x(n) = Z^{-1}[X(z)] = Z^{-1}[1 - z^{-1} + z^{-2}]$$

$$= \{\underset{\uparrow}{1}, -1, 1\}$$ **Ans.**

Example 7.57. Obtain the inverse z-transform of the following X(z) by the partial fraction expansion method

$$X(z) = \frac{z+2}{2z^2 - 7z + 3}$$

if the ROCs are

(i) $|z| > 3$ **(ii)** $|z| < \frac{1}{2}$ **(iii)** $\frac{1}{2} < |z| < 3$

Solution : We require the partial fraction expansion of $\frac{X(z)}{z}$.

Thus, we have

$$X'(z) = \frac{X(z)}{z} = \frac{z+2}{z(2z^2 - 7z + 3)} = \frac{z+2}{2z\left(z - \frac{1}{2}\right)(z-3)}$$

$$X'(z) = \frac{A_1}{z} + \frac{A_2}{\left(z - \frac{1}{2}\right)} + \frac{A_3}{(z-3)}$$ (using partial fraction expansion)

$$X'(z) = \frac{2}{3} - \frac{z}{z - \frac{1}{2}} + \frac{z/3}{z-3}$$ ...(i)

Here, the given function X(z) has two poles namely $p_1 = 1/2$ and $p_2 = 3$ and it would have the following three inverse z-transforms :

(i) In the region $|z| > 3$, all poles are interior, *i.e.*, the signal $x(n)$ is causal, and therefore,

$$x(n) = \frac{2}{3}\delta(n) - \left(\frac{1}{2}\right)^n u(n) + \frac{1}{3}(3)^n u(n)$$

(ii) In the region $|z| < \frac{1}{2}$, both the poles are exterior, *i.e.*, signal $x(n)$ is anticausal and therefore.

$$x(n) = \frac{2}{3}\delta(n) - \left(\frac{1}{2}\right)^n u(-n-1) - \frac{1}{3}(3)^n u(-n-1)$$

(*iii*) In the region, $\frac{1}{2} < |z| < 3$, the pole $p_1 = \frac{1}{2}$ is interior and $p_2 = 3$ is exterior, and therefore

$$x(n) = \frac{2}{3}\delta(n) - \left(\frac{1}{2}\right)^n u(n) - \frac{1}{3}(3)^n u(-n-1) \quad \textbf{Ans.}$$

Example 7.58. *Figure 7.14 shows a low-pass RC network. Find the equivalent discrete-time expressions for the circuit output responses $y(n)$, when the input $x(t) = e^{-2t}$ and the sampling frequency is $f_s = 50$ Hz.*

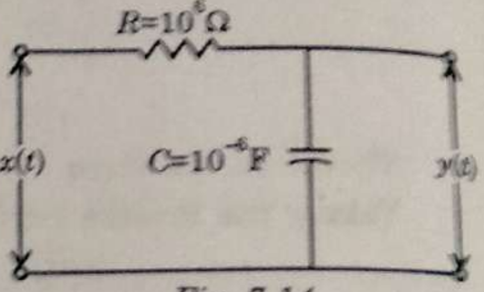

Fig. 7.14.

Solution : The transfer function of the given network in the s-domain may be expressed as under:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1/RC}{s + \frac{1}{RC}} = \frac{1}{s+1} \qquad ...(i)$$

Taking the inverse Laplace transform, we obtain

$$h(t) = \text{Inverse Laplace transform } \{H(s)\}$$

$$= \text{Inverse Laplace transform } \left\{\frac{1}{s+1}\right\} = e^{-t} \qquad ...(ii)$$

Now let us use z-transform approach.

Substituting $t = nT_s$ in equation (*ii*), we obtain

$$h(nT_s) = e^{-nT_s} \qquad ...(iii)$$

Taking the z-transform of equation (*iii*), we obtain

$$H(z) = Z\,[h(nT_s)] = Z\,[e^{-nT_s}]$$

$$= \frac{z}{z - e^{-T_s}} \qquad ...(iv)$$

We have given input

$$x(t) = e^{-2t} \qquad ...(v)$$

Substituting $t = nT_s$ in equation. (*v*), we obtain

$$x(nT_s) = e^{-2nT_s} \qquad ...(vi)$$

Taking z-transform of equation (*vi*), we obtain

$$X(z) = Z\,[x(nT_s)] = Z\,[e^{-2nT_s}]$$

$$X(z) = \frac{z}{z - e^{-2T_s}} \qquad ....(vii)$$

We know that $\quad Y(z) = H(z)\,X(z) \qquad ...(viii)$

Substituting equations (*iv*) and (*vii*) in equation (*viii*), we obtain

$$Y(z) = \frac{z}{(z - e^{-T_s})}\,\frac{z}{(z - e^{-2T_s})}$$

or $$\frac{Y(z)}{z} = \frac{z}{(z-e^{-T_s})(z-e^{-2T_s})}$$

$$\frac{Y(z)}{z} = \frac{A_1}{(z-e^{-T_s})} + \frac{A_2}{(z-e^{-2T_s})}$$

(using partial fraction expansion)

or $$\frac{Y(z)}{z} = \frac{\left(\frac{1}{1-e^{-T_s}}\right)}{(z-e^{-T_s})} + \frac{\left(\frac{1}{1-e^{T_s}}\right)}{(z-e^{-2T_s})}$$

or $$Y(z) = \frac{1}{(1-e^{-T_s})}\frac{z}{(z-e^{-T_s})} + \frac{1}{(1-e^{-T_s})}\frac{z}{(z-e^{-2T_s})} \quad ...(ix)$$

Taking inverse z-transform of equation (*ix*), we obtain

$$y(nT_s) = \frac{1}{(1-e^{-T_s})}(e^{-nT_s}) + \frac{1}{(1-e^{T_s})}(e^{-2nT_s}) \quad ...(x)$$

Substituting $T_s = \frac{1}{50} = 0.02$ in equation (*x*), we obtain

$$y(n) = 50.5(0.980)^n - 49.5(0.961)^n$$

Hence, the required output response will be

$$y(n) = 50.5(0.980)^n - 49.5(.0961)^n$$ **Ans.**

Example 7.59. Find the z-transform of the following functions :

(*i*) $x(n) = -a^n u(-n-1)$

(*ii*) $x(n) = a^{-n} u(-n-1)$

Solution : (*i*) We know that *Z*-transform is given by

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

or $$X(z) = -\sum_{n=1}^{\infty}(a^{-1}z)^n = -\sum_{n=0}^{\infty}(a^{-1}z)^n$$

Further, we know that

$$\sum_{n=0}^{\infty}(a^{-1}z)^n = \frac{1}{1-a^{-1}z} \quad \text{if } |a^{-1}z| < 1 \text{ or } |z| < |a|$$

Thus, we have $$X(z) = -\frac{1}{1-a^{-1}z} = \frac{-a^{-1}z}{1-a^{-1}z} = \frac{z}{z-a}$$

$$= \frac{1}{1-az^{-1}} \quad |z| < |a|$$

(*b*) Similarly, we have

$$X(z) = \sum_{n=-\infty}^{\infty} a^{-n} u(-n-1) z^{-n} = \sum_{n=-\infty}^{-1} (az)^{-n}$$

$$X(z) = \sum_{n=1}^{\infty}(az)^n = \sum_{n=0}^{\infty}(az)^n - 1$$

Again, we have

$$\sum_{n=0}^{\infty}(az)^n = \frac{1}{1-az} \quad \text{if } |az| < 1 \text{ or } |z| < \frac{1}{a}$$

Thus, we have $\quad X(z) = \frac{1}{1-az} - 1 = \frac{az}{1-az} = -\frac{z}{z-1/a} \quad |z| < \frac{1}{a}$ **Ans.**

Example 7.60. A finite sequence $x[n]$ is defined as under :

$$x(n) = \begin{cases} \neq 0 & N_1 \leq n \leq N_2 \\ = 0 & \text{otherwise} \end{cases}$$

where N_1 and N_2 are finite. Show that the ROC of $X(z)$ is the entire z-plane except possibly $z = 0$ or $z = \infty$.

Solution : We have

$$X(z) = \sum_{n=N_1}^{N_2} x(n)\, z^{-n} \quad \text{...(i)}$$

For z not equal to zero or infinity, each term in equation (*i*) will be finite and thus $X(z)$ will converge. If $N_1 < 0$ and $N_2 > 0$, then equation (*i*) includes terms with both positive powers of z and negative powers of z. As $|z| \to 0$, terms with negative powers of z become unbounded, as $|z| \to \infty$, terms with positive powers of z become unbounded. Hence, the ROC is the entire z-plane except for $z = 0$ and $z = \infty$. If $N_1 \geq 0$, equation (*i*) contains only negative powers of z, and hence the ROC includes $z = \infty$. If $N_2 \leq 0$, equation (*i*) contains only positive powers of z, and hence the ROC includes $z = 0$.

Example 7.61. Find the z-transform $X(z)$ and sketch the pole-zero plot with the ROC for each of the following sequences :

(a) $x(n) = \left(\frac{1}{2}\right)^n . u(n) + \left(\frac{1}{3}\right)^n . u(n)$

(b) $x(n) = \left(\frac{1}{3}\right)^n . u(n) + \left(\frac{1}{2}\right)^n . u(-n-1)$

(c) $x(n) = \left(\frac{1}{2}\right)^n . u(n) + \left(\frac{1}{3}\right)^n . u(-n-1)$

Solution : (*a*) We have

$$\left(\frac{1}{2}\right)^n u(n) \leftrightarrow \frac{z}{z-\frac{1}{2}} \quad |z| > \frac{1}{2} \quad \text{...(i)}$$

$$\left(\frac{1}{3}\right)^n u(n) \leftrightarrow \frac{z}{z-\frac{1}{3}} \quad |z| > \frac{1}{3} \quad \text{...(ii)}$$

It may be observed that the ROCs in equation (i) and (ii) overlap, and therefore, we have

$$X(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z-\frac{1}{3}} = \frac{2z\left(z-\frac{5}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)} \qquad |z| > \frac{1}{2} \qquad ...(iii)$$

From equation (iii), it may be observed that $X(z)$ has two zeros at $z = 0$ and $z = \frac{5}{12}$ and two poles at $z = \frac{1}{2}$ and $z = \frac{1}{3}$ and that the ROC is $|z| > \frac{1}{2}$, as sketched in figure 7.15(a).

(b) We also have

$$\left(\frac{1}{3}\right)^n u(n) \leftrightarrow \frac{z}{z-\frac{1}{3}} \qquad |z| > \frac{1}{3} \qquad ...(iv)$$

$$\left(\frac{1}{2}\right)^n u(-n-1) \leftrightarrow -\frac{z}{z-\frac{1}{2}} \qquad |z| < \frac{1}{2} \qquad ...(v)$$

It may be observed the ROCs in equations (iv) and (v) overlap, and thus, we have

$$X(z) = \frac{z}{z-\frac{1}{3}} - \frac{z}{z-\frac{1}{2}} = -\frac{1}{6}\frac{z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)} \qquad \frac{1}{3}<|z|<\frac{1}{2} \qquad ...(vi)$$

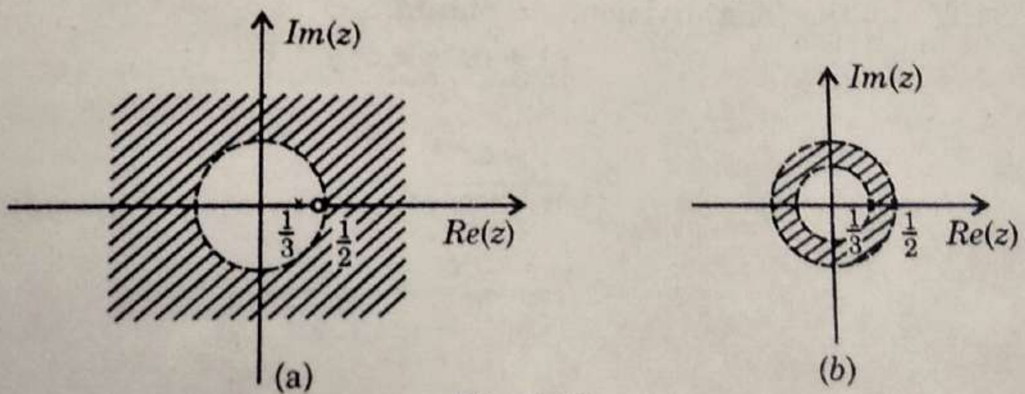

Fig. 7.15.

From equation (vi), it may be observed that $X(z)$ has one zero at $z = 0$ and two poles at $z = \frac{1}{2}$ and $z = \frac{1}{3}$ and that the ROC is $\frac{1}{3} < |z| < \frac{1}{2}$, as sketched in figure 7.15(b).

(c) In this case, we write

$$\left(\frac{1}{2}\right)^n u(n) \leftrightarrow \frac{z}{z-\frac{1}{2}} \qquad |z| > \frac{1}{2} \qquad ...(vii)$$

$$\left(\frac{1}{3}\right)^n u(-n-1) \leftrightarrow \frac{z}{z-\frac{1}{3}} \qquad |z| < \frac{1}{3} \qquad ...(viii)$$

It may be observed that the ROCs (vii) and ($viii$) do not overlap and that there is no common ROC and thus $x(n)$ will not have $X(z)$.

Example 7.62. Find the inverse z-transform of

$$X(z) = z^2\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})(1+2z^{-1}) \quad 0 < |z| < \infty \qquad ...(i)$$

Solution : We can express $X(z)$ as under :

$$X(z) = z^2 + \frac{1}{2}z - \frac{5}{2} + z^{-1}$$

Then, we have

$$X(z) = x(-2)z^2 + x(-1)z + x(0) + x(1)\,z^{-1}$$

and we get

$$x(n) = \left\{\ldots..,0,1,\frac{1}{2},\underset{\uparrow}{-\frac{5}{2}},1,0,\ldots..\right\} \quad \textbf{Ans.}$$

Example 7.63. Using the power series expansion technique, find the inverse z-transform of the following $X(z)$:

(a) $X(z) = \dfrac{1}{1-az^{-1}}, |z| > |a$

(b) $X(z) = \dfrac{1}{1-az^{-1}}, |z| < |a$

Solution : (a) Since the ROC is $|z| > |a|$, that is, the exterior of a circle, $x(n)$ is a right-sided sequence. Thus, we must divide to obtain a series in the power of z^{-1}. Carrying out the long division, we obtain.

$$\begin{array}{r|l} & 1 + az^{-1} + a^2 z^{-2} + \ldots \\ \hline 1 - az^{-1} & z \\ & \underline{1 - az^{-1}} \\ & \quad az^{-1} \\ & \quad \underline{az^{-1} - a^2 z^{-2}} \\ & \qquad a^2 z^{-2} \\ & \qquad \vdots \\ & \qquad \vdots \end{array}$$

Thus,

$$X(z) = \frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + \ldots + a^k z^{-k} + \ldots$$

and thus, we have

$$x(n) = 0 \quad n < 0$$
$$x(0) = 1$$
$$x(1) = a$$
$$x(2) = a^2 \ldots.$$
$$x(k) = a^k \ldots$$

Hence, we obtain

$$x(n) = a^n\,u(n) \quad \textbf{Ans.}$$

(b) Since the ROC is $|z| < |a|$ that is, the interior of a circle, $x(n)$ is a left-sided sequence. Thus, we must divide so as to obtain a series in the power of z as follows. Multiplying both the numerator and denominator or $X(z)$ by z, we have

$$X(z) = \frac{z}{z-a}$$

and carrying out the long division, we obtain

$$\begin{array}{r|l} -a+z & \overline{z} \quad -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - \ldots \\ & \underline{z - a^{-1}z^2} \\ & a^{-1}z^2 \\ & \underline{a^{-1}z^2 - a^{-2}z^3} \\ & a^{-2}z^3 \\ & \vdots \\ & \vdots \end{array}$$

Thus, $$X(z) = \frac{1}{1-az^{-1}} = -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - \ldots - a^{-k}z^k - \ldots$$

and thus, we have

$$x(n) = 0, \quad n \geq 0$$

$$x(-1) = -a^{-1}, \quad x(-2) = -a^{-2}, \quad x(-3) = -a^{-3} \ldots \qquad x(-k) = -a^{-k} \ldots$$

Thus, we get

$$x(n) = -a^n u(-n-1) \quad \textbf{Ans.}$$

Example 7.64. Find the inverse z-transform of the following $X(z)$:

(a) $X(z) = \log\left(\dfrac{1}{1-az^{-1}}\right), |z| > |a$

(b) $X(z) = \log\left(\dfrac{1}{1-az^{-1}}\right), |z| < |a$

Solution : (*a*) The power series expansion for $\log(1-r)$ is given by

$$\log(1-r) = -\sum_{n=1}^{\infty} \frac{1}{n} r^n \quad |r| < 1 \qquad \ldots(i)$$

Now, $$X(z) = \log\left(\frac{1}{1-az^{-1}}\right) = -\log(1-az^{-1}) \qquad |z| > |a|$$

Since the ROC is $|z| > |a|$, that is, $|az^{-1}| < 1$, $X(z)$ has the power series expansion as under :

$$X(z) = \sum_{n=1}^{\infty} \frac{1}{n}(az^{-1})^n = \sum_{n=1}^{\infty} \frac{1}{n} a^n z^{-n}$$

from which we can identify $x[n]$ as

$$x(n) = \begin{cases} (1/n)a^n & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

or $$x(n) = \frac{1}{n} a^n u[n-1] \quad \textbf{Ans.}$$

(b) Here, $X(z) = \log\left(\frac{1}{1-a^{-1}z}\right)$

or $X(z) = -\log(1 - a^{-1} z) \;\; |z| < |a|$

Since the ROC is $|z| < |a|$, that is, $|a^{-1} z| < 1$, therefore, $X(z)$ has the power series expansion as under :

$$X(z) = \sum_{n=1}^{\infty} \frac{1}{n}(a^{-1}z)^n$$

$$X(z) = \sum_{n=-1}^{-\infty} -\frac{1}{n}(a^{-1}z)^{-n} = \sum_{n=-1}^{-\infty} -\frac{1}{n} a^n z^{-n}$$

from which we can identify $x(n)$ as

$$x(n) = \begin{cases} 0 & n \geq 0 \\ -(1/n)a^n & n \leq -1 \end{cases}$$

or $$x(n) = -\frac{1}{n}a^n u(-n-1) \quad \textbf{Ans.}$$

Example 7.65 (a) Determine the system function $H(z)$ and the frequency response of the system whose impulse response is given as

$$h(n) = \frac{1}{2}\left[\left(\frac{1}{2}\right)^n + \left(-\frac{1}{4}\right)^n\right] u(n)$$

and locate zeros and poles in z-plain.

(U.P. Tech., sem Examination 2001-2002)

Solution : Given impulse response

$$h(n) = \frac{1}{2}\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n + \left(-\frac{1}{4}\right)^n\right] u(n)$$

We know that

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n}$$

or $$H(z) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n} + \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n z^{-n}$$

or $$H(z) = \frac{1}{2}\left[\sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n + \sum_{n=0}^{\infty}\left(-\frac{1}{4}z^{-1}\right)^n\right]$$

↓ ROC $|z| > \frac{1}{2}$ ↓ ROC $|z| > \frac{1}{4}$

Now, the combined ROC = $|z| > \frac{1}{2}$

Further, we have

$$H(z) = \frac{1}{2}\left[\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\left(-\frac{1}{4}z^{-1}\right)}\right]$$

or
$$H(z) = \frac{1}{2}\left[\frac{z}{\left(z-\frac{1}{2}\right)} + \frac{z}{\left(z+\frac{1}{4}\right)}\right] = \frac{1}{2}\left[\frac{z^2+\frac{z}{4}+z^2-\frac{z}{2}}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right]$$

or
$$H(z) = \frac{z^2-\frac{z}{8}}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)} = \frac{z\left(z-\frac{1}{8}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}$$

Poles of $H(z)$ will be at $z = \frac{1}{2}$ and $z = -\frac{1}{4}$

Zeros of $H(z)$ will be at $z = 0$ and $z = \frac{1}{8}$. Ans.

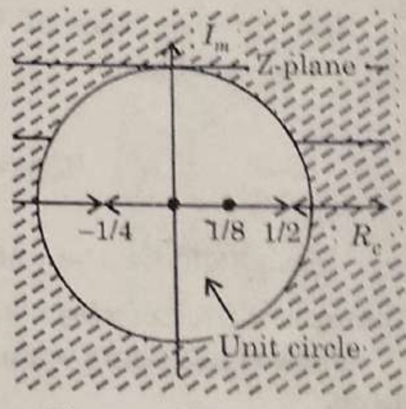

Fig. 7.16. Pole zero plot.

Example 7.66. (*i*) Determine the unit step response of the system described the following difference equation

$$\mathbf{y(n) = 0.9\,y(n-1) - 0.81y(n-2) + x(n)}$$

under the following initial condition:

$$\mathbf{y(-1) = y(-2) = 0}$$

(*ii*) Find the Z-transform of the following time functions:

1. **ramp function**
2. **impulse function**

(U.P. Tech., Sem Examination 2001-2002)

Solution : (*i*) Given that

$$y(n) = 0.9\,y(n-1) - 0.81\,y(n-2) + x(n) \quad ...(i)$$

Also, given y(−1) = y(−2) = 0

Takeing z transform of equation (*i*), we have

$$Y(z) = 0.9[z^{-1}\,Y(z) + y(-1)] - 0.81\,[z^{-2}\,Y(z) + z^{-1}\,y(-1) + y(-2)] + X(z)$$

$$Y(z) = 0.9[z^{-1}\,Y(z)] - 81[z^{-2}\,Y(z)] + X(z)$$

so
$$H(z) = \frac{Y(z)}{X(z)} = 1 - 0.9z^{-4} + 0.81z^{-2}$$

Therefore, impulse response will be given by

$$h[n] = \begin{cases} 0.1 & \text{for } n = 0 \\ -0.9 & \text{for } n = 1 \\ 0.81 & \text{for } n = 2 \\ 0 & \text{for } n = \text{otherwise} \end{cases}$$

Step response

$$s(n) = \sum_{k=-\infty}^{\infty} h(k) = \begin{cases} 0 & \text{for } K < 0 \\ 1 & \text{for } K = 0 \\ 0.1 & \text{for } K = 1 \\ 0.91 & \text{for } K \geq 2 \end{cases}$$

(*i*) Z-transform of Ramp function

Here, $x(n) = n,\ n \geq 0$

$$Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} n\, z^{-n}$$

$$X(z) = z^{-1} + 2z^{-2} + 3z^{-3} + \ldots$$

which is an infinite term G.P.

$$H(z) = \frac{z}{(z-1)^2} = \frac{z-1}{(1-z^{-1})^2}$$

Also, ROC $|z| > 1$

(*ii*) Z-transform of impulse function:

Here, we have

$$H(z) = Z[\delta(n)] = \sum_{n=-\infty}^{\infty} \delta(n)\, z^{-n} = 1$$

Here, ROC will be entire z-plane.

Example 7.67. Let $x(n) = 0,\ n < 0$ and $x(n) \leftrightarrow X(z)$ show that region of convergence of $X(z)$ must be exterior of some circle in z-plane. ***(U.P. Tech, Sem. Examination 2002-2003)***

Solution : Given $x(n) = 0,\ n < 0$

The above sequence is right sided sequence of infinite duration.

Now, since z-transform of such sequence is given by

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

Therefore, we have

$$x(n)z^{-n} = 0 \quad \text{for } n < 0$$
$$\neq 0 \quad \text{for } n \geq 0$$

On putting $z = r_1 e^{jQ}$, we have sequence $\sum_{n=0}^{\infty} x(n)\, (r_1 e^{jQ})^{-n}$

Thus, we write

$$X(z) = \sum_{n=0}^{\infty} x(n)\, r_1^{-n}\, e^{jQn}$$

↓

this is also summable

or $$\sum_{n=0}^{\infty} x(n)\, r_1^{-n} < \infty$$

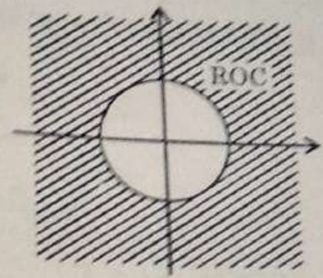

Fig. 7.17.

Thus, we have negative power of z as this is causal and right sided sequence which form an infinite term G.P series, whose sum can be expressed as $\left(\frac{a}{1-r}\right)$. The denominator will provide a pale so, ROC will be defined from the exterior to this circle upto ∞.

Example 7.68. Determine the z-transform of following sequence with region of convergence :

(i) $u(n)$

(ii) $-u(-n-1)$

What do you conclude from these z-transforms ?

(U.P. Tech, sem. Examination 2002-2003)

Solution : (i) $$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

Now, since $$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

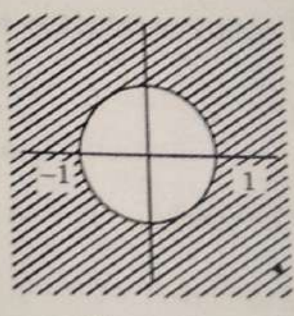

Fig. 7.18.

or $$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

or $$X(z) = \sum_{n=0}^{\infty} 1\, z^{-n}$$

$X(z)$ $$X(z) = \sum_{n=0}^{\infty} (z^{-1})^n$$

or $$X(z) = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots.$$

$$X(z) = \frac{1}{1-\frac{1}{z}}$$

Thus, $u(n) \xleftrightarrow{z} = \frac{z}{z-1}$, ROC : $|z| > 1$

Therefore, the ROC is defined for exterior to circle $z = 1$ (outside the unit circle)

(ii) Here, we have

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{-1} - u(-n-1)\, z^{-n} = -\sum_{n=-\infty}^{-1} z^{-n}$$

Putting $n = -m$, we have

$$X(z) = -\sum_{m=1}^{\infty} z^{m} = z + z^2 + z^3 + \ldots z^{\infty}$$

$$= \frac{-z}{1-z}$$

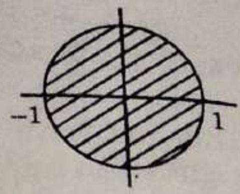

Fig. 7.19.

ROC : $|z| < 1$

Therefore, the ROC is interior of a circle having radius 1 (units circle)

Example 7.69. Derive the expression for the z-transform with ROC of a sequence which is convolution of two sequences. ***(U.P. Tech, Sem Examination 2002-03)***

Solution: The convolution property is one of the most important property of the z-transform because it is used to convert be convolution of two discrete-time signals in time domain into multiplication of their z-transform.

$$x(n) = x_1(n) \otimes x_2(n)$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \,.\, x_2(n-k)$$

From convolution property, we have

$$x_1(n) \xleftrightarrow{z} X_1(z)$$

$$x_2(n) \xleftrightarrow{z} X_2(z)$$

Therefore, $x(n) = x_1(n) \otimes x_2(n) \xleftrightarrow{z} X(z) = X_1(z).\, X_2(z)$

Since, $$x(n) = \sum_{k=-\infty}^{\infty} x_1(k) \,.\, x_2(n-k)$$

Therefore, $$X(z) = Z\{x(n)\} = \sum_{k=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \right] z^{-n}$$

or $$X(z) = \sum_{k=-\infty}^{\infty} x_1(k) \sum_{k=-\infty}^{\infty} x_2(n-k)\, z^{-n}$$

Now using time shifting property, we get

$$x(n - n_o) \xrightarrow{z} z^{-n_0}\, X(z)$$

Thus, we write

$$\sum_{m=-\infty}^{\infty} x(k) \left\{ \sum_{n=-\infty}^{\infty} x_2(n-k)\, z^{-n} \right\}$$

$= X_1(z) \,.\, X_2(z)$ **Hence Proved.**