Example 7.36. Find the z-transform and the ROC of the signal.

$$x(n) = \left[3(2^n) - 4(3^n)\right] \cdot u(n)$$

Solution : The given signal is

$$x(n) = \left[3(2^n) - 4(3^n)\right] \cdot u(n) \qquad ...(i)$$

Assuming that $x_1(n) = 2^n.u(n)$

and $x_2(n) = 3^n.u(n)$

Then the signal $x[n]$ may be written as

$$x(n) = 3x_1(n) - 4x_2(n) \qquad ...(ii)$$

According to linearity property, the z-transform of equation (ii) is

$$X(z) = 3X_1(z) - 4X_2(z)$$

Also, we know that

$$\alpha^n.u(n) \xleftrightarrow{z} \frac{1}{1-\alpha z^{-1}} \quad \text{ROC} : |z| > |\alpha| \qquad ...(iii)$$

Substituting $\alpha = 2$ and $\alpha = 3$ in equation (iii), we get

$$x_1(n) = 2^n.u(n) \xleftrightarrow{z} X_1(z)$$

or

$$x_1(n) = \frac{1}{1-2z^{-1}} \qquad \text{ROC} : |z| > 2$$

and

$$x_2(n) = 3^n.u[n] \xleftrightarrow{z} X_2(z)$$

or

$$x_2(n) = \frac{1}{1-3z^{-1}} \qquad \text{ROC} : |z| > 3$$

It may be observed that the intersection of the ROC of $X_1(z)$ and $X_2(z)$ is $|z| > 3$.

Therefore the overall z-transform $X(z)$ will be

$$X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}} \qquad \text{ROC} : |z| > 3$$

Example 7.37. Evaluate the z-transform of the following signal :

$$x(n) = \left(\frac{1}{2}\right)^n \cdot u(n)$$

Solution : The given signal is

$$x(n) = \left(\frac{1}{2}\right)^n \cdot u(n) \qquad ...(i)$$

We know that z-transform of a signal $x[n]$ is expressed as

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Putting the value of $x[n]$, we get

or

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cdot u(n)\, z^{-n} \qquad [\because u(n) \text{ exists only for positive } n]$$

or

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cdot 1.z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cdot z^{-n}$$

or $$X(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n = 1+\left(\frac{1}{2}z^{-1}\right)+\left(\frac{1}{2}z^{-1}\right)^2+\left(\frac{1}{2}z^{-1}\right)^3+\ldots\ldots \quad \ldots(ii)$$

The above expression is a well-known geometric progression of the form

$$1 + x + x^2 + x^3 + \ldots\ldots = \frac{1}{1-x} \qquad \text{if } |x| < 1$$

Therefore, the equation (*ii*) becomes

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad \text{for } \left|\frac{1}{2}z^{-1}\right| < 1 \text{ or } |z| > \frac{1}{2}$$

Hence $$X(z) = \frac{z}{z-\frac{1}{2}} \qquad \text{ROC} : |z| > \frac{1}{2}$$

Example 7.38. Find the *z*-transform of the signal

$$x(n) = -\alpha^n\, u(-n-1) = \begin{cases} 0 & n \geq 0 \\ -\alpha^n & n \leq -1 \end{cases}$$

Solution : The given signal is

$$x(n) = -\alpha^n\, u(-n-1) = \begin{cases} 0 & n \geq 0 \\ -\alpha^n & n \leq -1 \end{cases} \quad \ldots(i)$$

We know that *z*-transform is expressed as

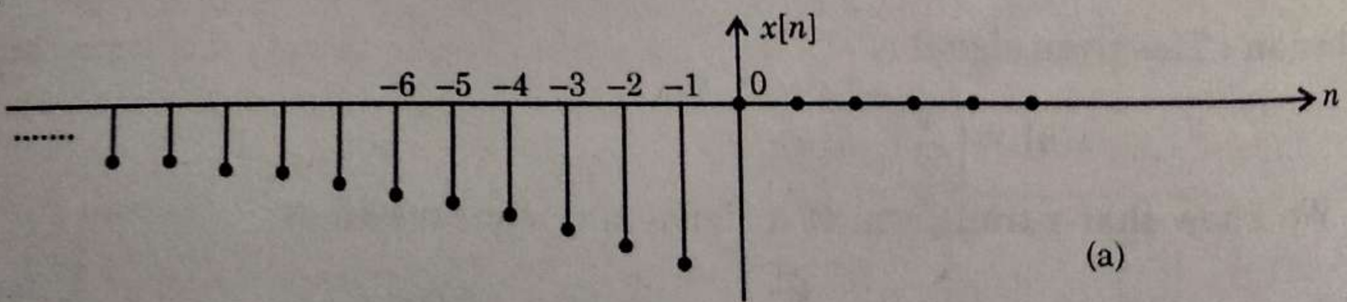
$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

According to equation (*i*) $$X(z) = \sum_{n=-\infty}^{\infty}(-\alpha^n)\, z^{-n}$$

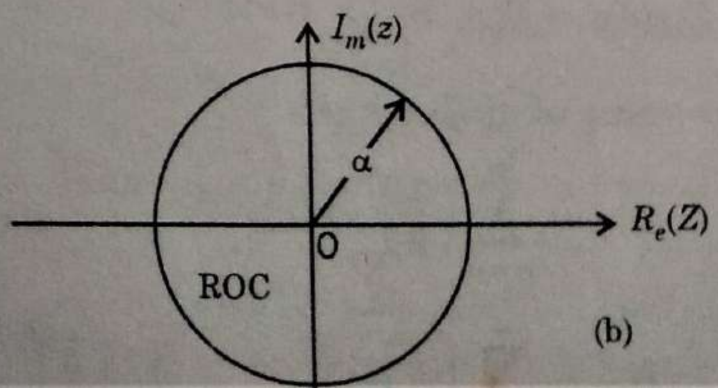
Putting $m = -n$ $$X(z) = -\sum_{m=1}^{\infty}(-\alpha^{-1} z)^m$$

Using the formula $$x + x^2 + x^3 + \ldots\ldots = \frac{x}{1-x} \qquad \text{for } |x| < 1$$

Therefore, We have $$X(z) = -\frac{\alpha^{-1} z}{1-\alpha^{-1} z}$$



(a)



(b)

$$X(z) = -\frac{1}{1-\alpha z^{-1}} \quad \text{for } |\alpha^{-1}z| < 1 \text{ or } |z| < |\alpha|$$

or

$$X(z) = -\frac{1}{1-\alpha z^{-1}} \quad \text{ROC : } |z| < |\alpha|$$

Example 7.39. Find the z-transform of the signal

$$\boldsymbol{x(n) = na^n.u(n)}$$

Solution : The given signal is

$$x(n) = na^n.u(n) \qquad ...(i)$$

The signal in equation (i) may be expressed as $nx_1(n)$

where $x_1(n) = a^n.u(n)$

But, we know that

$$x_1(n) = a^n.u(n) \xleftrightarrow{z} X_1(z) = \frac{1}{1-az^{-1}} \quad \text{ROC : } |z| > |a|$$

Also the differentation property states that

If $x(n) \xleftrightarrow{z} X(z)$

then $nx(n) \xleftrightarrow{z} -z \cdot \frac{dX(z)}{dz}$

Therefore, $na^n.u(n) \xleftrightarrow{z} X(z) = -z \cdot \frac{dX_1(z)}{dz} = \frac{az^{-1}}{(1-az^{-1})^2}$ ROC : $|z| > |a|$

Example 7.40. Find the discrete-time signal $x[n]$ whose z-transform is given as

$$\boldsymbol{X(z) = \log(1 + az^{-1}) \qquad |z| > |a|}$$

Solution : The given expression is

$$X(z) = \log(1 + az^{-1}) \qquad |z| > |a|$$

Taking the differentiation, we get

$$\frac{dX(z)}{dz} = \frac{-az^{-2}}{1+az^{-1}}$$

Hence

$$-z\frac{dX(z)}{dz} = az^{-1}\left[\frac{1}{1-(-a)z^{-1}}\right] \qquad |z| > |a| \qquad ...(i)$$

In equation (i), the inverse z-transform of the term in brackets is $(-a)^n$.

The multiplication by z^{-1} means a time-delay of one sample (*i.e.* time-shifting property) which results in $(-a)^{n-1}$. $u[n-1]$.

In last, from the differentiation property, we get

$$nx(n) = a(-a)^{n-1}.u(n-1)$$

or

$$x(n) = (-1)^{n+1}.\frac{a^n}{n} \cdot u(n-1) \qquad \textbf{Ans.}$$

Example 7.41. Find the pole-zero plot for the discrete-time causal signal

$$\boldsymbol{x(n) = a^n.u(n) \qquad a > 0}$$

Solution : We know that the z-transform is expressed as

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

or

$$X(z) = \sum_{n=-\infty}^{\infty} a^n \cdot u(n)\, z^{-n} = \sum_{n=0}^{\infty} a^n \cdot 1 \cdot z^{-n} \qquad [\because u[n] = 1 \text{ for } n > 0]$$

or $$X(z) = \sum_{n=0}^{\infty}(az^{-1})^n = \sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n$$

or $$X(z) = 1+\frac{a}{z}+\left(\frac{a}{z}\right)^2+\left(\frac{a}{z}\right)^3+\ldots\ldots = \frac{1}{1-\frac{a}{z}} \quad \left|\frac{a}{z}\right|<1 \text{ or } |z|>|a$$

or $$X(z) = \frac{z}{z-a} \qquad \text{ROC : } |z| > |a| \qquad \ldots(i)$$

From equation (*i*), it is clear that $X(z)$ has one zero at $z_1 = 0$ and one pole at $p_1 = a$.

Figure 7.12. shows the pole-zero plot.

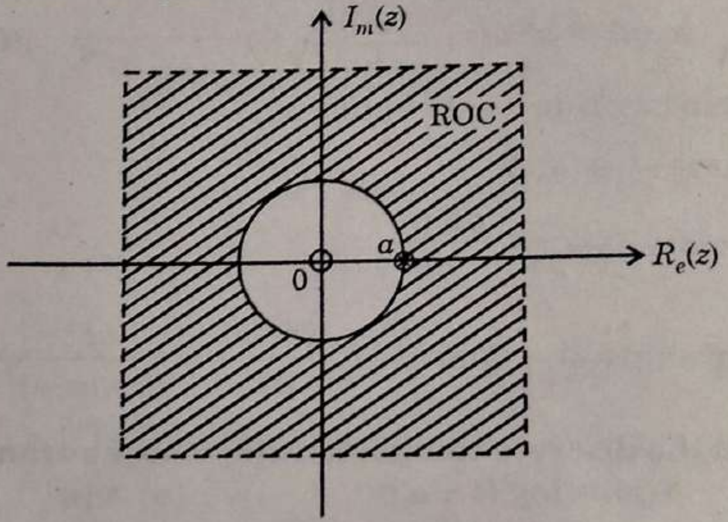


Fig. 7.12. Pole-zero plot for the causal exponential signal $x[n] = a^n.u[n]$.

It may be noted that the pole at $p_1 = a$ is not included in the region of the convergence (ROC) since the z-transform does not converge at a pole.

Example 7.42. Find the z-transform of the following sequences :

(*i*) $\quad x_1(n) = \{1, 2, 3, 4, 5, 0, 7\}$

(*ii*) $\quad x_2(n) = \{1, 2, 3, 4, 5, 0, 7\}$
$\qquad\qquad\qquad\qquad\uparrow$

Solution: (*i*) Let us consider the first sequence $x_1[n]$, Its samles are

$x_1(0) = 1 \qquad x_1(4) = 5$

$x_1(1) = 2 \qquad x_1(5) = 0$

$x_1(2) = 3 \qquad x_1(6) = 7$

$x_1(3) = 4$

The z-transform of this sequence may be obtained with the help of the expression

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n)\, z^{-n}$$

Here, $x_1[n]$ has the values from $n = 0$ to $n = 6$.
Thus, the above equation becomes

$$X_1(z) = \sum_{n=0}^{6} x_1(n)\, z^{-n}$$

Substituting the values of $x_1(n)$ and expanding, we get

$$X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4} + 0.\,z^{-5} + 7\,z^{-6}$$

or
$$X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4} + 7z^{-6}$$

or
$$X_1(z) = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^4} + \frac{7}{z^6}$$

Now, let us determine the region of convergence for this z-transform. $X_1(z)$ has finite values except at $z = 0$. At $z = 0$, $X_1(z)$ becomes infinite.

Hence $X_1(z)$ is convergent for all values of z except at $z = 0$.

Therefore, the region of convergence (ROC) will be:

ROC : Entire z-plance except $z = 0$.

(*ii*) Let us consider the second sequence $x_2[n]$. Its samples are

$$x_2(-3) = 1 \qquad x_2(1) = 5$$
$$x_2(-2) = 2 \qquad x_2(2) = 0$$
$$x_2(-1) = 3 \qquad x_2(3) = 7$$
$$x_2(0) = 4$$

Here, $x_2(n)$ has the values from $n = -3$ to $n = 3$.

Thus, z-transform will be expressed as

$$X_2(z) = \sum_{n=-3}^{3} x_2(n)\, z^{-n}$$

Substituting the values of $x_2[n]$ in above equation and expanding, we get

$$X_2(z) = 1.\, z^3 + 2z^2 + 3z^1 + 4z^{\circ} + 5z^{-1} + 0.z^{-2} + 7z^{-3}$$

or
$$X_2(z) = z^3 + 2z^2 + 3z + 4 + 5z^{-1} + 7z^{-3}$$

or
$$X_2(z) = z^3 + 2z^2 + 3z + 4 + \frac{5}{z} + \frac{7}{z^3}$$

Now let us determine the region of convergence (ROC) for this z-transform. Here, $X_2(z)$ is infinite for $z = 0$ and $z = \infty$. The z-transform expressed by above equation is not convergent at $z = 0$ and $z = \infty$. Hence the region of convergence will be :

ROC : Entire z-plane except $z = 0$ and $z = \infty$. Ans.

Example 7.43. Find the z-transform and ROC of the following sequence :

$$x(n) = \left(-\frac{1}{3}\right)^n .u(n) - \left(\frac{1}{2}\right)^n .u(-n-1)$$

Solution : The z-transform of $x[n]$ will be given as

$$Z\{x(n)\} = X(z) = Z\left\{\left(-\frac{1}{3}\right)^n .u(n) - \left(\frac{1}{2}\right)^n .u(-n-1)\right\}$$

since z-transform satisfies linearity property, therefore we may write above equation as

$$X(z) = Z\left\{\left(-\frac{1}{3}\right)^n .u(n)\right\} + Z\left\{-\left(\frac{1}{2}\right)^n .u(-n-1)\right\} \qquad (i)$$

Here let us use the standard results derived earlier *i.e.*,

$$Z\{a^n.\, u(n)\} = \frac{1}{1-az^{-1}},\ ROC: |z| > \frac{1}{3}$$

Therefore, $$Z\left\{\left(-\frac{1}{3}\right)^n .u(n)\right\} = \frac{1}{1+\frac{1}{3}z^{-1}},\ ROC: |z| > \frac{1}{3} \qquad ...(ii)$$

Also, $$Z\{-a^n.\, u(-n-1)\} = \frac{1}{1-az^{-1}}\ ROC: |z| < |a$$

Therefore, $$Z\left\{-\left(\frac{1}{2}\right)^n .u(-n-1)\right\} = \frac{1}{1-az^{-1}}\ ROC: |z| < \frac{1}{2} \qquad (iii)$$

Now, using equations (*ii*) & (*iii*), the equation (*i*) may be written as

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}};\ \text{ROC}: |z| > \frac{1}{3} \text{ and } |z| < \frac{1}{2}$$

The ROC of $z| > \frac{1}{3}$ and $|z| < \frac{1}{2}$ may also be written as $\frac{1}{3} < |z| < \frac{1}{2}$. Hence, ROC is the region between the circles of radius of $\frac{1}{3}$ and $\frac{1}{2}$. Figure 7.14 shows this ROC

Therefore, the ROC is the intersection or simply the overlap of ROC of individual functions. It is annular region for $\frac{1}{3} < |z| < \frac{1}{2}$ looking like a disk. **Ans.**

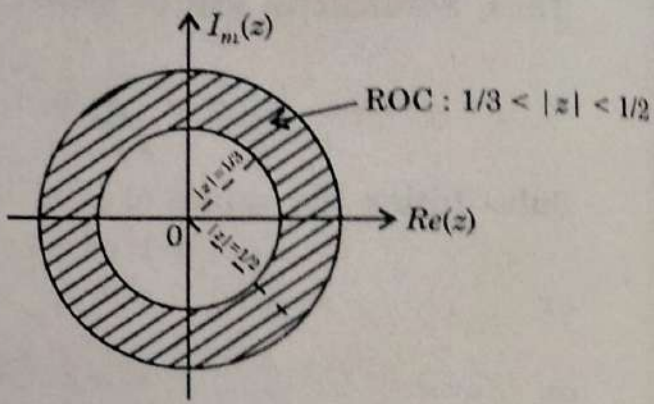


Fig. 7.13. ROC of the function of the example 7.44.

Example7.45. Solve the difference equation of a causal discrete-time LTI system which is expressed as under :

$$y(n) + 3\,y(n) = x(n)$$

Assume that the system is initially relaxed.

Solution : We know that the transfer function $H(z)$ can be determined by taking unilateral z-transform of both sides of equation (*i*)

$$y(n) + 3\,y(n-1) = x(n) \qquad ...(i)$$

$$Z\,[y(n) + 3y(n-1)] = Z[x(n)]$$

or $$Y(z) + 3z^{-1}\,Y(z) = X(z) \qquad ...(ii)$$

The system is initially relaxed, *i,e.*, all the initial conditions are zero for this system,

Using equation (*ii*), we get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{\left(1+3z^{-1}\right)} \qquad ...(iii)$$

This is the transfer function of the causal LTI system described by equation (*i*)

Now, let us assume that input is given by

$$x(n) = u(n) \qquad ...(iv)$$

Taking z-transform of equation (*iv*), we obtain

$$X(z) = U(z) = \frac{1}{1-z^{-1}} \qquad ...(v)$$

Further, we know that

$$Y(z) = H(z)\,X(z) = \frac{1}{(1+3z^{-1})(1-z^{-1})}$$

Now, Using partial fraction expression, we get

$$Y(z) = \frac{1}{(1+3z^{-1})(1-z^{-1})} = \frac{\alpha_1}{1+3z^{-1}} + \frac{\alpha_2}{(1-z^{-1})}$$

or

$$Y(z) = \frac{3/4}{(1+3z^{-1})} + \frac{1/4}{(1-z^{-1})} \qquad ...(vi)$$

Taking inverse unilateral z-transform of both sides of equation (*iv*), we obtain

$$Z^{-1}[Y(z)] = Z^{-1}\left[\frac{3/4}{1+3z^{-1}} + \frac{1/4}{1-z^{-1}}\right] = \frac{3}{4}Z^{-1}\left[\frac{1}{1+3z^{-1}}\right] + \frac{1}{4}Z^{-1}\left[\frac{1}{1-z^{-1}}\right]$$

or

$$y(n) = \frac{3}{4}(-3)^n\,u(n) + \frac{1}{4}(1)^n\,u(n)$$

$$y(n) = \left[\frac{1}{4} + \frac{3}{4}(-3)^n\right]u(n)$$

This is the required solution for given difference equation. **Ans.**

Example 7.45. Find the step response of a system which is expressed by

$$y(n) = Ay(n-1) + x(n), \quad -1 < A < 1$$

When the initial condition is y(–1) = 1.

Solution: By taking one-sided z-transform of both sides of the given difference equation, we get

$$Y(z) = A[z^{-1}\,Y(z) + y(-1)\,] + X(z) = A\,[z^{-1}\,Y(z) + 1] + X(z)$$

or

$$Y(z)\,[1 - Az^{-1}] = A + X(z) \qquad ...(i)$$

However, for step response, $x(n) = u(n)$

Then, we have

$$X(z) = Z\,[\,u(n)\,] = \frac{1}{1-z^{-1}} \qquad ...(ii)$$

Now, substituting equation (*ii*) in equation (*i*), we obtain

$$Y(z)\,[\,1 - Az^{-1}] = A + \frac{1}{1-z^{-1}}$$

or

$$Y(z) = \frac{A}{1-Az^{-1}} + \frac{1}{(1-z^{-1})(1-Az^{-1})} \qquad ...(iii)$$

Using partial fraction expansion, we get

$$Y(z) = \frac{A}{1-Az^{-1}} + \frac{B_1}{1-z^{-1}} + \frac{B_2}{1-Az^{-1}} \qquad ...(iv)$$

$$1 - z^{-1} = 0 \qquad B_1 = \frac{1}{1 - Az^{-1}} = \frac{1}{1 - A}$$

or $$z = 1$$

$$1 - A^{z-1} = 0 \qquad B_2 = \frac{1}{1 - z^{-1}} = \frac{1}{1 - \frac{1}{A}} = \frac{A}{A - 1} = -\frac{A}{1 - A}$$

or $$z^{-1} = 1/A$$

Substituting the value of B_1 and B_2 in equation (*iv*), we get

$$y(z) = \frac{A}{1 - Az^{-1}} + \frac{\left(\frac{1}{1-A}\right)}{1 - z^{-1}} + \frac{\left(\frac{-A}{1-A}\right)}{1 - Az^{-1}} \qquad ...(v)$$

Taking the inverse z-transform of equation (*v*), we get

$$y(n) = A^{n+1}u(n) + \frac{1}{(1-A)}u(n) + \frac{-1}{(1-A)}A^{n+1}u(n)$$

or $$y(n) = A^{n+1}\, u(n) + \left[\frac{1 - A^{n+1}}{1 - A}\right]u(n)$$

or $$y(n) = \frac{1}{1-A}[1 - A^{n+2}]\,u(n) \quad \textbf{Ans.}$$

Example 7.46. Solve the following difference equation by using *z*-transform method

$$x(n + 2) + 3x(n + 1) + 2x(n) = 0$$

Given that initial conditions are $x(0) = 0$ and $x(1) = 1$.

Solution: Given difference equation is

$$x(n + 2) + 3x(n + 1) + 2x(n) = 0 \qquad ...(i)$$

Taking the z-transform of both sides of the above equation, we obtain

$$[z^2X(z) - z^2x(0) - zx(1)] + 3[zX(z) - z(0)] + 2\,[X(z)] = 0$$

or $$[z^2X(z) - z^2(0) - z(1)] + 3\,[zX(z) - z(0)] + 2\,[X(z)] = 0$$

or $$[z^2X(z) - z] + 3zX(z) + 2X(z) = 0$$

or $$X(z)\,[z^2 + 3z + 2] = z$$

or $$X(z) = \frac{z}{z^2 + 3z + 2} \qquad ...(ii)$$

Let us now take the inverse *z*-transform of above *z*-transform by partial fraction expansion method, *i.e.*,

$$X(z) = \frac{z}{z^2 + 3z + 2} = \left\{\frac{A}{(z+1)} + \frac{B}{(z+2)}\right\}$$

or $$X(z) = \frac{z}{z+1} - \frac{z}{z+2} = \frac{1}{1+z^{-1}} - \frac{1}{1+2z^{-1}} \qquad ...(iii)$$

or $$X(z) = \frac{1}{1+z^{-1}} - \frac{1}{1+2z^{-1}} \qquad ...(iv)$$

Taking the inverse z-transform of equation (*iv*), we obtain

$$x(n) = Z^{-1}\left[\frac{1}{1+z^{-1}}\right] - Z^{-1}\left[\frac{1}{1+2z^{-1}}\right]$$

or $$x(n) = (-1)^n u(n) - (-2)^n u(n) \quad \textbf{Ans.}$$

Example 7.47. Determine the response of the following system:

$$\boldsymbol{x(n+2) - 3\,x(n+1) + 2\,x(n) = \delta(n)}$$

Assume that all the initial conditions are zero.

Solution: Given system is

$$x(n+2) - 3\,x(n+1) + 2\,x(n) = \delta(n) \quad ...(i)$$

Taking one-sided z-transform of both sides of above equation, we obtain

$$z^2X(z) - 3\,z^1X(z) + 2\,X(z) = 1$$

or $$X(z)\,[z^2 - 3z + 2] = 1$$

or $$X(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-2)(z-1)}$$

$$= \frac{A}{z-2} + \frac{B}{z-1} = \frac{1}{z-2} - \frac{1}{z-1} \quad ...(ii)$$

(By partial fraction expansion)

Taking inverse z-transform of both sides of equation (*ii*), we have

$$x(n) = Z^{-1}\left[\frac{1}{z-2}\right] - Z^{-1}\left[\frac{1}{z-1}\right] = Z^{-1}\left[\frac{z^{-1}}{1-2z^{-1}}\right] - Z^{-1}\left[\frac{z^{-1}}{1-z^{-1}}\right]$$

$$x(n) = (2)^{n-1} - (1)^{n-1}$$

or $$x(n) = -1 - (2)^{n-1} \quad \textbf{Ans.}$$

Example 7.48. Determine the z-transform of the following discrete-time signals. Also find the ROC for each of the following cases:

(*i*) $x(n) = 2^n\,u(n) + 3\left(\frac{1}{2}\right)^n u(n)$ **(*ii*)** $x(n) = 3\left(-\frac{1}{2}\right)^n u(n) - 2\,(3)^n\,u(-n-1)$

Solution: (*i*) Given signal is

$$x(n) = 2^n\,u(n) + 3\left(\frac{1}{2}\right)^n u(n)$$

We know that Two-sided (bilateral) z-transform of $x(n)$ is defined as

$$X(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\,z^{-n} = \sum_{n=-\infty}^{\infty}\left[2^n\,u(n) + 3\left(\frac{1}{2}\right)^n u(n)\right] z^{-n}$$

or $$X(z) = \sum_{n=-\infty}^{\infty} 2^n u(n)\,z^{-n} + 3\sum_{n=-\infty}^{\infty}\left(\frac{1}{2}\right)^n u(n)\,z^{-n}$$

or $$X(z) = \sum_{n=0}^{\infty} 2^n\,(1)\,z^{-n} + 3\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n (1)\,z^{-n} \quad \left[\text{Since } u(n) = \begin{cases}1, & \text{for } n \ge 0\\ 0, & \text{for } n < 0\end{cases}\right]$$

or $$X(z) = \sum_{n=0}^{\infty}(2z^{-1})^n + 3\sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n$$

or $$X(z) = \underbrace{\frac{1}{(1-2z^{-1})}}_{\text{I-part}} + \underbrace{3\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)}}_{\text{II-part}}$$

or $$X(z) = \frac{\left(1-\frac{1}{2}z^{-1}\right)+3(1-2z^{-1})}{(1-2z^{-1})\left(1-\frac{1}{2}z^{-1}\right)} = \frac{4-13z^{-1}}{(1-2z^{-1})\left(1-\frac{1}{2}z^{-1}\right)} \quad ...(i)$$

Now, **ROC of I-part :** $|1-2z^{-1}| > 0$

or $$|2z^{-1}| < 1 \text{ or } |z^{-1}| < \frac{1}{2}$$

or $$\left|\frac{1}{z}\right| < \frac{1}{2} \text{ or } |z| > 2 \quad ...(ii)$$

ROC of II-part : $$\left|1-\frac{1}{2}z^{-1}\right| > 0 \text{ or } \left|\frac{1}{2}z^{-1}\right| < 1$$

or $$|z^{-1}| < 2 \text{ or } \left|\frac{1}{z}\right| < 2 \text{ or } |z| > \frac{1}{2} \quad ...(iii)$$

Therefore, ROC of $X(z)$, will be

$$\text{ROC}: \{|z| > 2\} \cap \left\{|z| > \frac{1}{2}\right\}$$

Thus, we have $$\text{ROC}: \{|z| > 2\}$$

(*ii*) Given signal is $x(n) = 3\left(-\frac{1}{2}\right)^n u(n) - 2(3)^n u(-n-1)$

We know that Two-sided z-transform of $x(n)$ is given by

$$X(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty}\left[3\left(-\frac{1}{2}\right)^n u(n) - 2(3)^n u(-n-1)\right] z^{-n}$$

or $$X(z) = 3\sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^n u(n)\, z^{-n} - 2\sum_{n=-\infty}^{\infty}(3)^n u(-n-1)\, z^{-n}$$

or $$X(z) = 3\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n (1)\, z^{-n} - 2\sum_{n=-\infty}^{-1}(3)^n (1)\, z^{-n}$$

$$\left[\text{Since } u(n) = \begin{cases}1, & n \geq 0\\ 0, & n < 0\end{cases} \text{ and } u(-n-1) = \begin{cases}1, & (-n-1) \geq 0 \to n \leq -1\\ 0, & (-n-1) < 0 \to n > -1\end{cases}\right]$$

or $$X(z) = 3\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} - 2\sum_{n=-\infty}^{-1}(3)^n z^{-n}$$

or $$X(z) = 3\sum_{n=0}^{\infty}\left(-\frac{1}{2}z^{-1}\right)^n - 2\sum_{n=-\infty}^{-1}(3z^{-1})^n$$

$$= 3\sum_{n=0}^{\infty}\left(-\frac{1}{2}z^{-1}\right)^n - 2\sum_{n=1}^{\infty}(3z^{-1})^{-n}$$

or $$X(z) = 3\frac{1}{1-\left(-\frac{1}{2}z^{-1}\right)^n} - 2\frac{\left(3z^{-1}\right)^{-1}}{1-(3z^{-1})^{-1}}$$

$$= \frac{3}{1+\frac{1}{2}z^{-1}} - \frac{2/3z^{-1}}{1-\frac{1}{3z^{-1}}} = \frac{3}{1+\frac{1}{2}z^{-1}} - \frac{2}{3z^{-1}-1}$$

or $$X(z) = \underset{\text{I-part}}{\frac{3}{1+\frac{1}{2}z^{-1}}} - \underset{\text{II-part}}{\frac{2}{1-3z^{-1}}} \quad ...(iv)$$

or $$X(z) = \frac{2(1-3z^{-1})+2\left(1+\frac{1}{2}z^{-1}\right)}{\left(1+\frac{1}{2}z^{-1}\right)(1-3z^{-1})} = \frac{4-5z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)(1-3z^{-1})}$$

Now, ROC of I-part : $\left|1+\frac{1}{2}z^{-1}\right| < 0$ or $\left|\frac{1}{2}z^{-1}\right| < 1$

or $$z^{-1}| < 2 \quad \text{or} \quad z| > \frac{1}{2} \quad ...(v)$$

Also, ROC of II-part : $\left|1-3z^{-1}\right| > 0$ or $\left|3z^{-1}\right| > 1$ or $z| < 3$ ...(vi)

Therefore, ROC of $X(z)$ $\quad \text{ROC}: \left\{|z| > \frac{1}{2}\right\} \cap \{|z| < 3\} = \frac{1}{2} < |z| < 3$ **Ans.**

Example 7.49. Two discrete-time signals are given as under :

$$x_1(n) = \left(\frac{1}{3}\right)^n u(n) \text{ and } \qquad x_2(n) = \left(\frac{1}{5}\right)u(n)$$

Convolution of $x_1(n)$ and $x_2(n)$ is given by

$$x(n) = x_1(n) \otimes x_2(n)$$

(i) Find $X(z)$ using convolution property of z-transform.

(ii) Find $x(n)$ by taking inverse z-transforms of $X(z)$ by using partial fraction expansion method.

Solution : (a) Given that :

$$x(n) = x_1(n) \otimes x_2(n)$$

From convolution property of z-transform, we know

$$x(n) = x_1(n) \otimes x_2(n) \xleftrightarrow{Z} X(z) = X_1(z)\, X_2(z)$$

or $\quad X(z) = X_1(z)\,X_2(z)$ ...(i)

where $\quad X_1(z) = Z[x_1(n)] = Z\left[\left(\frac{1}{3}\right)^n u(n)\right] = \frac{1}{1-\frac{1}{3}z^{-1}}, ROC : |z| > \frac{1}{3}$. ...(ii)

and $\quad X_2(z) = Z[x_2(n)] = Z\left[\left(\frac{1}{5}\right)^n u(n)\right] = \frac{1}{1-\frac{1}{5}z^{-1}}, \text{ROC} : |z| > \frac{1}{5}$ ...(iii)

Substituting equations (*ii*) and (*iii*) in equation (*i*), we obtain

$$X(z) = X_1(z)\,X_2(z)$$

$$= \frac{1}{\left(1-\frac{1}{3}z^{-1}\right)}\frac{1}{\left(1-\frac{1}{5}z^{-1}\right)} = \frac{1}{\left(1-\frac{1}{3}z^{-1}\right)}\frac{1}{\left(1-\frac{1}{5}z^{-1}\right)} \quad ...(iv)$$

$$\text{ROC} : \left\{|z| > \frac{1}{3}\right\} \cap \left\{|z| > \frac{1}{5}\right\} = z| > \frac{1}{3}$$

(*ii*) Using equation (*iv*), we have

$$X(z) = \frac{z^2}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{5}\right)}$$

or $$\frac{X(z)}{z} = \frac{z}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{5}\right)} = \frac{A_1}{\left(z-\frac{1}{3}\right)} + \frac{A_2}{\left(z-\frac{1}{5}\right)}$$

(using partial fraction expansion

or $$\frac{X(z)}{z} = \frac{5/2}{\left(z-\frac{1}{3}\right)} + \frac{-3/2}{\left(z-\frac{1}{5}\right)} = \frac{5/2z}{\left(z-\frac{1}{3}\right)} - \frac{3/2z}{\left(z-\frac{1}{5}\right)}$$

$$\frac{X(z)}{z} = \frac{5/2}{\left(1-\frac{1}{3}z^{-1}\right)} - \frac{3/2}{\left(1-\frac{1}{5}z^{-1}\right)}$$

Note that since both the poles are surrounded by ROC, therefore, they gi only positive time sequence terms.

Now, taking inverse z-transform of both sides of equation (*iii*), we have

$$x(n) = Z^{-1}[X(z)] = Z^{-1}\left[\frac{5/2}{\left(1-\frac{1}{3}z^{-1}\right)} - \frac{3/2}{\left(1-\frac{1}{5}z^{-1}\right)}\right]$$

or $$x(n) = \frac{5}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{3}{2}\left(\frac{1}{5}\right)^n u(n) \quad \textbf{Ans.}$$

Example 7.50. Find the z-transform of the following sequence :

$$x(n) = \begin{cases} 2^n, & n < 0 \\ (1/2)^n, & n = 0, 2, 4 \\ (1/3)^n, & n = 1, 3, 5 \end{cases}$$

Solution : The given sequence is

$$x(n) = \begin{cases} 2^n, & n < 0 \\ (1/2)^n, & n = 0, 2, 4 \\ (1/3)^n, & n = 1, 3, 5 \end{cases} \qquad ...(i)$$

We know that z-transform of $x(n)$ is given by

$$X(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad ...(ii)$$

Substituting equation (i) in equation (ii), we obtain

$$X(z) = \sum_{n=-\infty}^{-1} 2^n z^{-n} + \sum_{\substack{n=0 \\ (n-\text{even})}}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{\substack{n=0 \\ (n-\text{odd})}}^{\infty} \left(\frac{1}{3}\right)^n z^{-n}$$

or
$$X(z) = \sum_{m=1}^{\infty} 2^{-m} z^m + \sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^{2p} z^{-2p} + \sum_{q=0}^{\infty} \left(\frac{1}{3}\right)^{2q-1} z^{-(2q+1)}$$

where $m = -n$, $p = \frac{n}{2}$ and $q = \left(\frac{n-1}{2}\right)$

Thus, we have

$$X(z) = X_1(z) + X_2(z) + X_3(z)$$

or
$$X(z) = \frac{z/2}{\left(1 - \frac{z}{2}\right)} + \frac{z^2}{\left(z^2 - \frac{1}{4}\right)} + \frac{z/3}{\left(z^2 - \frac{1}{9}\right)} \qquad ...(iii)$$

Now, ROC for $X_1(z) : |z| < 2$

ROC for $X_2(z) : |z| > \frac{1}{2}$

ROC for $X_3(z) : |z| > \frac{1}{3}$

Therefore, ROC for $X(z) : \{|z| < 2\} \cap \left\{|z| > \frac{1}{2}\right\} \cap \left\{|z| > \frac{1}{3}\right\}$

$$= \frac{1}{2} < |z| < 2 \quad \textbf{Ans.}$$

Example 7.51. Determine the convolution $x(n)$ of the following two signals

$$x_1(n) = \{4, \underset{\uparrow}{-2}, 1\}$$

$$x_2(n) = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{elsewhere} \end{cases}$$

Solution : Here, we have

$$X_1(z) = Z[x_1(n)] = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n}$$