6.10. Introduction to Laplace Transform

In previous chapters, we have discussed some mathematical tools such as Fourier series & Fourier transform to analyse signals and systems. The laplace transform is another mathematical tool which is used for the analysis of signals and systems. In fact, the Laplace transform provides broader characterization of the signals and systems compared to Fourier transform. In some cases, Laplace transform can be used where Fourier transform cannot be used. Laplace transform can be used for the analysis of unstable systems whereas Fourier transform has several limitations.

An important difference between the Fourier transform and the Laplace transform is that the Fourier transform uses a summation of waves of positive and negative frequencies whereas the Laplace transform employs damped waves through the use of an additional factor $e^{-\sigma}$ where σ is a positive number. Both the Fourier transform and the Laplace transform are mathematical operations which convert the time-domain function $x(t)$ to the frequency-domain function $X(e^{j\omega})$ and $X(s)$ respectively. Also, the Laplace Transform provides the total solution to the differential equation and the corresponding initial and final value problems.

The Laplace transform is an important and powerful mathematical tool in the system analysis and design. Laplace transform is widely used for describing the continuous circuits and systems including automatic control systems and also to analyse signal flow through the causal linear-time invariant systems with non-zero initial conditions. Also the z-transform, to be discussed in the next chapter is suitable for dealing with discrete signal and systems.

6.11. Definition of Laplace Transform

For periodic or non-periodic time-functions $x(t)$ which is zero for $t \leq 0$ and defined for $t >$, the laplace transform of $x(t)$ denoted as £ $\{x(t)\}$, may be defined as

$$\text{£}\{x(t)\} = X(s) = \int_0^{\infty} x(t)\, e^{-(\sigma + j\omega)\,t}\, dt \qquad \text{...(6.43)}$$

Putting $s = \sigma + j\omega$, we have

$$X(s) = \int_0^{\infty} x(t)\, e^{-st}\, dt$$

It may be noted that the integration is taken from 0 to ∞. Therefore, this is called **unilateral laplace transform.**

The unilateral laplace transform is mainly useful for the analysis of causal signals.

Similarly, the bilateral transform is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \qquad \text{...(6.44)}$$

Here, note that the integration is taken from $-\infty$ to $+\infty$. Thus it is called **bilateral or two sided laplace transform.**

The functions $x(t)$ ans $X(s)$ form a laplace transform pair *i.e.*,.

$$x(t) \xleftrightarrow{LT} X(s) \qquad \text{...(6.45)}$$

Here $x(t)$ = continuous-time signal

$X(s)$ = Laplace transform of $x(t)$

and s = complex variable = $\sigma + j\omega$

were σ and ω are the real and imaginary parts of the complex variable s respectively.

6.12. Relationship Between Laplace and Fourier Transforms

The laplace transform is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma - j\omega)t} dt$$

or

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-\sigma t} . e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} [x(t)\, e^{-\sigma t}]\, e^{-j\omega t}\, dt \qquad \text{...(6.46)}$$

The above equation shows that $X(s)$ is basically the Fourier transform of $x(t)e^{-\sigma t}$. Thus, we can say that the laplace transform of signal $x(t)$ may be assumed as the continuous-time Fourier transform of $x(t)$ after multiplication by a real exponential signal $e^{-\sigma t}$.Depending upon the value of σ, this real exponential signal

$e^{-\sigma t}$, may be growing or decaying with time. If σ is positive valued then the real exponential signal $e^{-\sigma t}$ will be decaying and if σ is negative valued then the real exponential signal $e^{-\sigma t}$ will be growing.

Now let us put $s = j\omega$ (*i.e.*, $\sigma = 0$) in equation (6.46)

then
$$X(s) = X(e^{j\omega}) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad ...(6.47)$$

or simply
$$X(e^{j\omega}) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

which is the continuous-time Fourier transform (CTFT) of a signal $x(t)$. Thus, we conclude

$$\text{CTFT}\,\{x(t)\} = X(s)\big|_{at\ s = j\omega} = LT\{x(t)\}_{at\ s = j\omega} \qquad ...(6.48)$$

This is the relationship between continuous time Fourier transform (CTFT) and Laplace transform.

Condition of Existence

The Laplace transform is defined as

$$£\,\{X(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \qquad ...(6.49)$$

The condition for the laplace transform to exist is

$$\int_{-\infty}^{\infty} \left| x(t)\, e^{-st} \right| dt < \infty \qquad ...(6.50)$$

for some finite values of σ.

Laplace transform thus converts the time-domain signal $x(t)$ to the frequency-domain signal $X(s)$. The transform defined to the positive-time functions is called single-sided or unilateral because it does not depend upon the history of $x(t)$ prior to $t = 0$.

In the double-sided or bilateral Laplace transform, the lower limit of integration is $t = -\infty$ where $x(t)$ covers all the time limits.

Thus, due to the convergence factor $e^{-\sigma t}$, the ramp, parabolic signals etc. are Laplace transformable. In transient problems, the Laplace transform is preferred to the continuous-time Fourier transform (CTFT) since the Laplace transform directly takes into account the initial conditions at $t = 0$, due to the lower limit of integration.

Inverse Laplace Transform

The inverse Laplace transform is used to convert frequency-domain signal $X(s)$ to the time-domain signal $x(t)$.

Mathematically,

$$x(t) = £^{-1}\,\{X(s)\} = \frac{1}{2\pi} \int_{\sigma_0 - j\omega}^{\sigma_0 + j\omega} X(s)\, e^{st}\, ds \qquad ...(6.51)$$

Thus, here the path of integration is simply a straight line parallel to the $j\omega$–axis, such that all the poles of $X(s)$ lie to the left of the line.

However, in practice, it is not necessary to carry out this complicated integration. With the help of partial fractions and existing tables of transform pairs we can obtain the solution of different equations.

6.13. The Region of Convergence (ROC)

As we know that for the existence of the Laplace transform, the integral

$$X(s) = \int_0^{\infty} x(t)\, e^{-st}\, dt \qquad ...(6.52)$$

must converge.

This limits the variable $s = \sigma + j\omega$ to a part of the complex plane known as the **Region of convergence (ROC)**.

As an example, let us take

$$X(s) = \pounds\{e^{3t}\} = \frac{1}{s-3} \qquad ...(6.53)$$

Here, the Laplace transform is defined only for $Re(s) > 3$. The region $Re(s) > 3$, *i.e.*, $\sigma > 3$ is called the region of convergence (ROC) as shown in figure 6.32.

In fact, Region of convergence (ROC) is required in computing the Laplace transform and the inverse Laplace transform. If the Region of convergence (ROC) is not specified, the inverse Laplace transform is not unique.

Also, in the one-sided Laplace transform, all time-functions are assumed to be positive and hence there is a one-to-one correspondence between the Laplace transform and its inverse laplace transform.

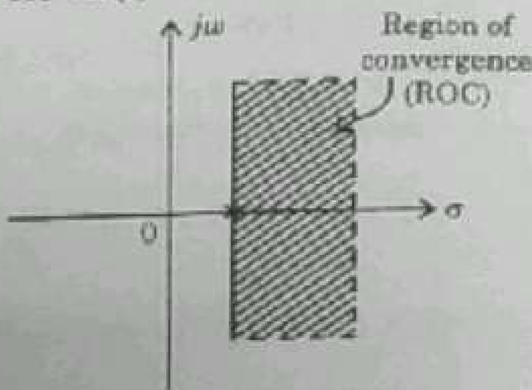

Fig. 6.32. Region of convergence for X(s) = 1/s − 3)

Therefore, no ambiguity will arise even if the Region of convergence (ROC) is not specified in the one-sided Laplace transform. However, in the two-sided Laplace transform, the specification of region of convergence (ROC) is essential.

6.13.1. Poles and Zeros of X(s)

Usually, $X(s)$ will be a rational function in s, *i.e.*,

$$X(s) = \frac{a_0 s^m + a_1 s^{m-1} + ... + a_m}{b_0 s^n + b_1 s^{n-1} + ... + b_n} = \frac{a_0 (s - z_1)...(s - z_m)}{b_0 (s - p_1)...(s - p_n)} \qquad ...(6.54)$$

The coefficients a_k and b_k are real constants, and m and n are positive integers. The $X(s)$ is called a **proper rational** function if $n > m$, and an **improper** rational function if $n \le m$. The roots of the numerator polynomial z_k, are called the zeros of X(s) because $X(s) = 0$ for those values of s. Similarly, the roots of the denominator polynomial, p_k, are called the poles of $X(s)$ because $X(s)$ is infinite for those values of s. Therefore, the poles of $X(s)$ lie outside the ROC since $X(s)$ does not converge at the poles, by definition. The zeros, on the other hand, may lie inside or outside the ROC. Except for a scale factor a_0/b_0, $X(s)$ can be completely specified by its zeros and poles. Thus, a

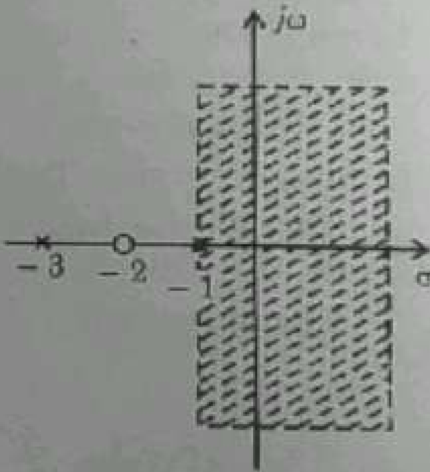

Fig. 6.33. s-plane representation of X(s) = (2s + 4)/(s² + 4s + 3)

very compact representation of $X(s)$ in the s-plane is to show the locations of poles and zeros in addition to the ROC.

Traditinally an "x" is used to indicate each pole location and an "0" is used to indicate each zero. This has been illustrated in figure 6.33 for $X(s)$ given by

$$X(s) = \frac{2s+4}{s^2+4s+3} = 2\frac{s+2}{(s+1)(s+3)} \qquad Re\,(s) > -1$$

It may be noted that $X(s)$ has one zero at $s = -2$ and two poles at $s = 1$ and $s = -3$ with scale factor 2.

6.13.2. Properties of Region of Convergence (ROC)

As a matter of fact the ROC of $X(s)$ depends on the nature of $x(t)$. The properties of the ROC are summarized below, assume that $X(s)$ is a rational function of s.

Property 1: The ROC does not contain any poles.

Property 2 : If $x(t)$ is a finite-duration signal, that is, $x(t) = 0$ except in a finite interval $t_1 \le t \le t_2$ $(-\infty < t_1$ and $t_2 < \infty)$, then the ROC is the entire s-plane except possibly $s = 0$ or $s = \infty$.

Property 3: If $x(t)$ is a right-sided signal, that is, $x(t) = 0$ for $t < t_1 < \infty$, then the ROC is of the form

$$Re(s) > \sigma_{max}$$

where σ_{max} equals the maximum real part of any of the poles of $X(s)$. Thus, the ROC is half-palne to the right of the vertical line $Re(s) = \sigma_{max}$ in the s-plane and thus to the right of all of the poles of $X(s)$.

Property 4: If $x(t)$ is a left-sided signal, that is, $x(t) = 0$ for $t > t_2 > -\infty$, then the ROC is of the form

$$Re(s) < \sigma_{min}$$

where σ_{min} equals the minimum real part of any of the poles of $X(s)$. Thus, the ROC is a half-plane to the left of the vertical line $Re(s) = \sigma_{min}$ in the s-plane and thus to the left of all of the poles of $X(s)$.

Property 5 : If $x(t)$ is a two sided signal, that is $x(t)$ is an infinite-duration signal that is neither right-sided nor left-sided, then the ROC is of the form

$$\sigma_1 < Re(s) < \sigma_2$$

where σ_1 and σ_2 are the real pats of the two poles of $X(s)$. Thus, the ROC is a vertical strip in the s-plane between the vertical lines $Re(s) = \sigma_1$ and $Re(s) = \sigma_2$.

Example 6.25. Determine the Laplace transform of a continuous-time signal given below:

$$x(t) = -e^{-at} \cdot u(-t)$$

Solution: We know that the Laplace transform of a signal $x(t)$ is expressed as

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

Putting the value of $x(t)$, we get

$$X(s) = \int_{-\infty}^{\infty} [-e^{-at}.u(-t)]\,.\,e^{-st}\, dt = -\int_{-\infty}^{\infty} e^{-at}.u(-t)\, e^{-st}\, dt \qquad ...(i)$$

Now

$$u(-t) = \begin{cases} 1 & \text{for } t \le 0 \\ 0 & \text{for } t > 0 \end{cases} \qquad ...(ii)$$

Substituting equation (*ii*) in equation (*i*) we get

$$X(s) = \int_{-\infty}^{0} e^{-at} e^{-st}\, dt = -\int_{-\infty}^{0} e^{-(s+a)t}\, dt$$

or
$$X(s) = \left[\frac{1}{-(s+a)} \cdot e^{-(s+a)t}\right]_{-\infty}^{0} = \frac{1}{s+a}$$

or
$$X(s) = \frac{1}{s+a},\ s+a<0 = \frac{1}{s+a},\ s<-a$$

Thus, the Laplace transform of a continuous-time signal $x(t) = -e^{-at} \cdot u(-t)$ may be written as

$$-e^{-at} \cdot u(-t) \xleftrightarrow{L.T} \frac{1}{s+a} \text{ for } \sigma < -a$$

σ is the real part of the complex variable

Example 6.26. Find the Laplace transform of a continuous-time signal given below:

$$x(t) = 2\, e^{-3t} \cdot u(t) - e^{-2t} \cdot u(t)$$

Also, determine the Region of convergence (ROC) of Laplace transform of this signal.

(Karnataka University, 1986)

Solution : We know that the Laplace transform of a signal $x(t)$ is defined as

$$X(s) = £\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

Putting the value of $x(t)$, we get

$$X(s) = \int_{-\infty}^{\infty} [2\, e^{-3t} \cdot u(t) - e^{-2t} \cdot u(t)]\, e^{-st}\, dt$$

or
$$X(s) = 2\int_{-\infty}^{\infty} e^{-3t} \cdot u(t)]\, e^{-st}\, dt - \int_{-\infty}^{\infty} e^{-2t} \cdot u(t)\, e^{-st}\, dt \qquad ...(i)$$

The unit-step signal $x(t)$ is given as

$$u(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} \qquad ...(ii)$$

using equations (*i*) & (*ii*), we get

$$X(s) = 2\int_{0}^{\infty} e^{-3t} e^{-st}\, dt \int_{0}^{\infty} e^{-2t} e^{-st}\, dt$$

$$= 2\int_{0}^{\infty} e^{-(s+3)t}\, dt - \int_{0}^{\infty} e^{-(s+2)t}\, dt$$

or $$X(s) = 2\left[\frac{1}{-(s+3)} e^{-(s+3)t}\right]_0^{\infty} - \left[\frac{1}{-(s+2)} e^{-(s+2)t}\right]_0^{\infty}$$

$$= \frac{1}{s+3} - \frac{1}{s+2}$$

This equation represents the Laplace transform of the signal

$$x(t) = 2\, e^{-3t} \cdot u(t) - e^{-2t} \cdot u(t)$$

Now, let us determine the Region of convergence (ROC) for the given function as under :

If $\quad e^{-3t} \cdot u(t) \xleftrightarrow{LT} \frac{1}{s+3}, \; R_1 : \sigma > -3$

and $\quad e^{-2t} \cdot u(t) \xleftrightarrow{LT} \frac{1}{s+3}, \; R_2 : \sigma > -2$

then $\; 2\, e^{-3t} \cdot u(t) - e^{-2t} \cdot u(t) \xleftrightarrow{LT} \frac{2}{s+3} - \frac{1}{s+1}$

$$= \frac{(s-1)}{(s+3)\,(s-1)}, \quad R = R_1 \cap R_2$$

$$= \sigma > -2 \quad \textbf{Ans.}$$

Here R_1, R_2 and R are the ROCs.

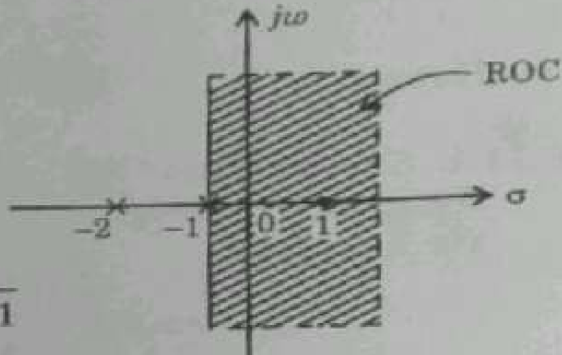

Fig. 6.34. Region of convergence (ROC) of the laplace transform of signal x(t)