7.11 STEP RESPONSE OF DISCRETE-TIME SYSTEMS

Definition The step response, $g(n)$, of an LSI system is defined as the response of the system to a unit-step sequence $u(n)$ given as input to the system when it is in ground state.

Hence,

$$g(n) = h(n)*u(n) = \sum_{k=-\infty}^{\infty} h(k)u(n-k)$$

But, $u(n-k) = 0$ for $k > n$ and is equal to 1, for $k \leq n$

$$\therefore \quad \boxed{g(n) = \sum_{k=-\infty}^{n} h(k)} \qquad \ldots(7.17)$$

This is analogous to the relation we got between $g(t)$ and $h(t)$ in the case of continuous-time LTI systems. There, we had

$$g(t) = \int_{-\infty}^{t} h(\lambda)d\lambda$$

The only difference is that the integral is replaced by a summation.

Also, from Eq.(7.17), we have:

$$g(n) - g(n-1) = \sum_{k=-\infty}^{n} h(k) - \sum_{k=-\infty}^{n-1} h(k) = h(n)$$

$$\therefore \quad \boxed{h(n) = g(n) - g(n-1)} \qquad \ldots(7.18)$$

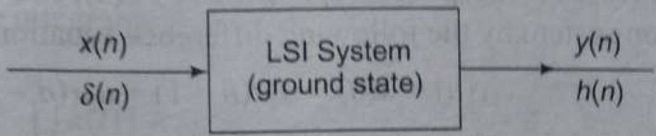
Since $g(n) - g(n-1)$ is discrete-time differentiation of $g(n)$, we find that Eq.(7.18) is analogous to the relation

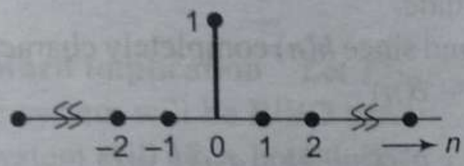
$$h(t) = \frac{dg(t)}{dt},$$

which we had for the LTI systems.

Unit-Sample Response of Causal Systems

Let T be a causal LSI system with $h(n)$ as its unit-sample response sequence. Let T be in ground state. Hence there can be no output till we apply some input, since, being causal, the system cannot anticipate an input and give an output. Let us apply a unit-sample sequence $\delta(n)$ to it as its input. The resulting output is what we call as the unit-sample response sequence and denote it by $h(n)$.





Since the input is zero for $n < 0$, and since the system is in ground state and is causal, the output has to be zero for $n < 0$.

$\therefore$ $\boxed{h(n) = 0 \text{ for } n < 0 \text{ for a causal system}}$

Example 7.24 *Find the step response of an LSI system whose unit-sample response is given by $h(n) = (0.5)^n u(n)$.*

Solution

$$h(k) = (0.5)^k u(k)$$

$$\therefore \quad g(n) = \text{step response} = \sum_{k=-\infty}^{n} (0.5)^k u(k)$$

$$= \sum_{k=0}^{n} (0.5)^k = \left[\frac{1-(0.5)^n}{1-0.5}\right] u(n) = 2\left[1-(0.5)^n\right] u(n).$$

Example 7.25 *A discrete-time LSI system is described by the following difference equation relating its input and output.*

$$y(n) - ay(n-1) = 2x(n)$$

Find the unit-sample response and step-response of the system.

Solution Let us determine the unit-sample response $h(n)$, first. Hence, put $x(n) = \delta(n)$, take the Z-Transform of the given difference equation and assume all initial conditions to be zero. We therefore get

$$Y(z) - a\left[z^{-1}Y(z) - y(0^-)\right] = 2Z[\delta(n)] = 2$$

$$\therefore \quad Y(z) = \frac{2z}{z-a} \text{ and } y(n) = h(n) = 2(a)^n u(n).$$

$\therefore$ step response $g(n) = \sum_{k=-\infty}^{n} 2a^k u(k) = 2\sum_{k=0}^{n} a^k = 2\left[\frac{a^n - 1}{a-1}\right] u(n)$

Example 7.26 *In this example, we shall try to model a multi-path communication system as an LSI system.*

Solution Multipath communication is one in which a signal arrives at the receiver by more than one path. Signals traversing paths other than the direct path arrive at the receiver after some delay and are generally weak compared to the one arriving via the direct path. For the sake of modeling the multipath propagation, let us, for the sake of simplicity, take only two more paths in addition to the direct path.

If the signal received via the direct path is represented by $x(n)$, and the total signal by $y(n)$, we may model the multipath propagation system by the following difference equation

$$y(n) = x(n) + a_1 x(n-1) + a_2 x(n-2)$$

where, a_1 and a_2 are constants, less than one in magnitude.

This difference equation represents an LSI system and since $h(n)$ completely characterizes an LSI system, we shall determine its $h(n)$. For this purpose, put $x(n) = \delta(n)$

$$\therefore \quad y(n)\big|_{x(n)=\delta(n)} = h(n) = \delta(n) + a_1\delta(n-1) + a_2\delta(n-2)$$

7.12 STABILITY

Although there are several ways of defining 'stability' of a system, one of the simplest and a very useful one at that, is the 'B-I-B-O stability', i.e., the 'Bounded-Input, Bounded-output' stability.

Before we state the BIBO criterion for stability, it is desirable to have a clear idea of what we mean by a 'bounded input' or a 'bounded output'. Since both input and output of the system are nothing but signals, we shall define the term, 'bounded signal'.

In general, a signal is said to be a bounded signal if its absolute value never exceeds some finite positive real number. If that finite positive real number is M, we say that the signal is bounded to a value M.

Hence, a continuous-time signal, $x(t)$ is a bounded signal, bounded to a value M, a finite positive real number, provided $|x(t)| \le M$ for all 't', i.e $-\infty < t < \infty$.

Similarly, a discrete-time signal, $x(n)$, is a bounded signal, bounded to a value M, a finite positive real number, provided $|x(n)| \le M$ for all n, i.e., $-\infty < n < \infty$.

BIBO Criterion for Stability of a System

A system T is said to be stable in the BIBO sense, provided, every bounded input given to the system results in a bounded output.

The reader should note that the word 'every', used in defining the criterion, is very important. It says that every bounded input given to the system must lead to a bounded output. That is, failure of the system to qualify itself to be called as a stable system for one particular bounded input signal is adequate to brand it as an unstable system, even if it had passed the test earlier with a million other input signals.

Earlier we had stated that since $h(t)$ and $h(n)$ completely characterize LTI and LSI systems respectively, they must have, embedded in them in some manner, information about the various properties of systems like causality, stability, etc. So, some feature of $h(t)$ (or $h(n)$) must be able to tell us whether or not a given LTI (or LSI) system is stable in the BIBO sense. In what follows, we shall examine this aspect in detail, first for the continuous-time LTI systems and then for the discrete-time LSI systems.

BIBO Stability of Continuous-time Systems

The condition imposed on the impulse response, $h(t)$, of an LTI system by the BIBO stability of the system, is stated in the form of a theorem, which is given below.

Theorem An LTI system T is stable in the BIBO sense *iff* $h(t)$, the impulse response of T is absolutely integrable.

As the reader might be aware, the word, '*iff*', stands for '*if and only if*'. Hence there are two implications.

(i) **The forward implication** An LTI system T is stable in the BIBO sense if $h(t)$, the impulse response of T, is absolutely integrable.

(ii) **The Reverse implication** An LTI system T cannot be stable in the BIBO sense unless $h(t)$, its impulse response function, is absolutely integrable.

Note $h(t)$ is said to be absolutely integrable if

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty, \quad i.e \int_{-\infty}^{\infty} |h(t)|\,dt \text{ is finite.}$$

(i) **Proof for forward implication** Let T be an LTI system with $h(t)$ as its impulse response function.

To prove that the system will be BIBO stable if $h(t)$ is absolutely integrable, we have to show that the output $y(t)$ of the system will be a bounded signal for every bounded input signal, if $h(t)$ is absolutely integrable.

Hence, let $x(t)$ be any arbitrary bounded signal, bounded to a finite positive real number M. Let $x(t)$ be given as input to the LTI system T. Then we know that its output $y(t)$ at any instant t, is given by the convolution of $x(t)$ with $h(t)$.

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau \qquad \ldots (7.19)$$

$$\therefore \quad |y(t)| = \left|\int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau\right| \le \int_{-\infty}^{\infty} |x(t-\tau)h(\tau)|\,d\tau$$

But
$$\int_{-\infty}^{\infty} |x(t-\tau)h(\tau)|\,d\tau = \int_{-\infty}^{\infty} |x(t-\tau)||h(\tau)|\,d\tau$$

and since $|x(t)| \le M$ for all t, we may replace $|x(t-\tau)|$ by its maximum possible value, M, and write

$$\int_{-\infty}^{\infty} |x(t-\tau)h(\tau)|\,d\tau = \int_{-\infty}^{\infty} |x(t-\tau)||h(\tau)|\,d\tau \le M\int_{-\infty}^{\infty} |h(\tau)|\,d\tau$$

$$\therefore \quad |y(t)| \le M\int_{-\infty}^{\infty} |h(\tau)|\,d\tau$$

Since M is a finite positive number, and $\int_{-\infty}^{\infty} |h(\tau)|\,d\tau$ is also a finite positive number, $|y(t)|$ is less than or equal to some finite positive number for any t. $\therefore$ $y(t)$ is also bounded. Since we have taken $x(t)$ to be any bounded signal, it means that any bounded signal given as input to the system T gives rise to a bounded signal as the output. Hence the system T is stable in the BIBO sense if its impulse response $h(t)$ is absolutely integrable.

(ii) Proof for the reverse implication Let T be an LTI system and let $h(t)$ be its impulse response. Now, we have to prove that T cannot be stable in the BIBO sense, if $h(t)$ is not absolutely integrable. As mentioned earlier, to show that a system is not stable, it is enough if we show that for a particular bounded input signal the system is giving an output which is not bounded. Again, to show that a signal is not bounded, it is enough if we can show that its value is not finite at some single instant. Hence, let us take an input signal $x(t)$ defined as follows

$$x(\tau) = \begin{cases} 1 & \text{if} \quad h(-\tau) > 0 \\ -1 & \text{if} \quad h(-\tau) < 0 \\ 0 & \text{if} \quad h(-\tau) = 0 \end{cases} \tag{7.20}$$

Obviously, such an $x(t)$ is a bounded signal because it can have an absolute value of either 1 or 0 at any instant, and is therefore bounded to a value 1.

When $x(t)$ is given as input, let $y(t)$ be the response of the system T. Then, we know that

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$\therefore \quad y(0) = y(t)\Big|_{t=0} = \int_{-\infty}^{\infty} x(\tau)\, h(-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} |h(-\tau)|\, d\tau \quad \text{from Eq. (7.20)}$$

$$\therefore \quad y(0) = \int_{-\infty}^{\infty} |h(-\tau)|\, d\tau = \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$$

But we are given that $h(t)$ is not absolutely integrable.

That is, $\int_{-\infty}^{\infty} |h(\tau)|\, d\tau \not< \infty$

$\therefore$ $y(0)$ is not finite. This means that $y(t)$ is not a bounded signal even though $x(t)$ is. Thus, T is not stable in the BIBO sense. Hence,

An LTI system with impulse response $h(t)$ is BIBO stable *iff* $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

BIBO Stability of LSI Systems

Theorem An LSI system T is stable in the BIBO sense *iff* its unit-sample response sequence $h(n)$ is absolutely summable, i.e, *iff* $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$.

The above theorem may be proved exactly in the same way as we did in the case of the continuous-time systems, and is therefore left as an exercise to the reader.

An LSI system with USR sequence $h(n)$ is stable in BIBO sense *iff* $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$