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Class: Signal & System
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Properties of Z-transform

7 Initial Value Theorem:

Statement: If $x(n)$ is a causal sequence then its initial value is given by:

$$x(0) = \lim_{Z \to \infty} X(Z).$$

Proof: Acc to definition of Z-transform

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad \text{--- (1)}$$

But if $x(n)$ is a causal sequence then limits of summation will be from $n=0$ to $n=\infty$

$$\therefore X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n} \quad \text{--- (2)}$$

Expanding the summation we get

$X(Z) = x(0) Z^{0} + x(1) Z^{-1} + x(2) Z^{-2} + x(3) Z^{-3} + \ldots$

Applying limits as $Z \to \infty$ we get

$\lim_{Z \to \infty} X(Z) = \lim_{Z \to \infty} x(0) + \lim_{Z \to \infty} x(1) Z^{-1}$

$+ \lim_{Z \to \infty} x(2) Z^{-2} + \ldots \quad (\text{as } Z^{0} = 1)$

$\therefore \lim_{Z \to \infty} X(Z) = \lim_{Z \to \infty} x(0) + \lim_{Z \to \infty} x(1) \cdot \frac{1}{Z} + \lim_{Z \to \infty} x(2) \frac{1}{Z^2} + \ldots$ (3)

Now as $Z \to \infty$ the terms $\frac{1}{Z} = \frac{1}{Z^2}$ etc will be $\frac{1}{\infty}$ which is zero thus eq. (3) becomes

$$\lim_{Z \to \infty} X(Z) = x(0).$$

But $x(0)$ is called as initial value of $x(z)$

Initial value $= x(0) = \lim_{Z \to \infty} X(Z)$

3 Final value Theorem:

Statement: If $x(n) \xrightarrow{Z} X(z)$ then

$$x(\infty) = \lim_{z \to 1} [(z-1) \cdot X(z)]$$

Proof:

Example: Find initial value and final value of $x(n)$ if z-transform is;

$$X(z) = \frac{2}{z^2 + \frac{1}{6}z - \frac{1}{6}}$$

Sol: Acc to initial value Theorem;

$$x(0) = \lim_{z \to \infty} X(z) = \lim_{z \to \infty} \frac{2}{z^2 + \frac{1}{6}z - \frac{1}{6}}$$

$$\therefore \quad x(0) = 0$$

Acc to final value Theorem

$$x(\infty) = \lim_{z \to 1} (z-1) \cdot X(z)$$

$$= \lim_{z \to 1} (z-1) \cdot \frac{2}{z^2 + \frac{1}{6}z - \frac{1}{6}}$$

We will obtain roots of denominator terms

$$z^2 + \frac{1}{6}z - \frac{1}{6} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-\frac{1}{6} \pm \sqrt{\frac{1}{36} + \frac{4}{6}}}{2}$$

$$\therefore \text{roots} = \frac{-\frac{1}{6} \pm \sqrt{\frac{1+24}{36}}}{2} = \frac{-\frac{1}{6} \pm \frac{5}{6}}{2}$$

$$\therefore \text{roots} = \frac{1}{3}, \frac{-1}{2}$$

$\therefore x(\infty) =$ …

$$\therefore x(\infty) = \lim_{z \to 1} (z-1) \cdot \frac{2}{\left(z - \frac{1}{3}\right)\left(z + \frac{1}{2}\right)}$$

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Ex. 3.3.31 : Find the z transform and region of convergence for the following sequence.

$$X(n) = 7\left(\frac{1}{3}\right)^n u(n) - 6\left(\frac{1}{2}\right)^n u(n)$$

Apply initial value theorem and check the z transform whether it is correct or not.

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Soln. :

Given : $$x(n) = 7\left(\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(n)$$

We have, $$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{z}{z-\alpha}, \quad |z| > |\alpha|$$

$$\therefore \quad \left(\frac{1}{3}\right)^n u(n) \overset{Z}{\longleftrightarrow} \frac{z}{z-\frac{1}{3}}, |z| > \frac{1}{3}$$

and $$\left(\frac{1}{2}\right)^n u(n) \overset{Z}{\longleftrightarrow} \frac{z}{z-\frac{1}{2}}, |z| > \frac{1}{2}$$

$$\therefore \quad X(z) = 7 \cdot \frac{z}{z-\frac{1}{3}} - 6 \cdot \frac{z}{z-\frac{1}{2}}$$

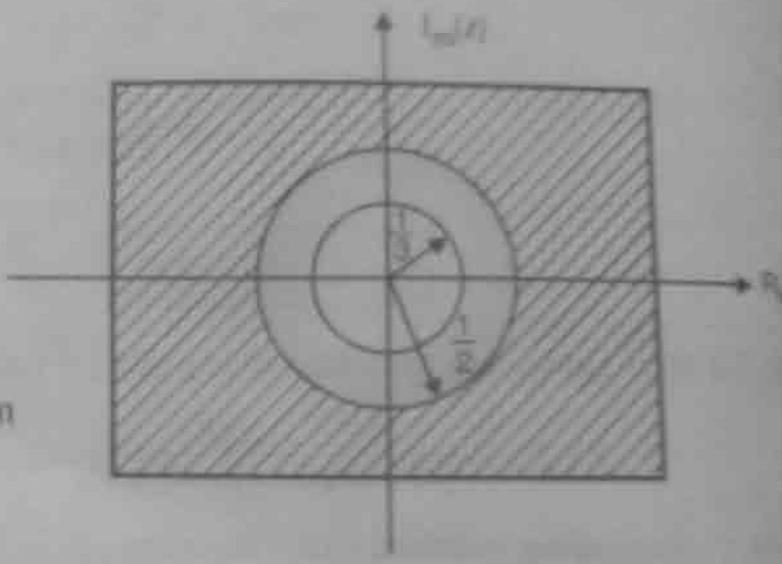

Fig. P. 3.3.31

The combined ROC is $|z| > \frac{1}{2}$. It is shown in Fig. P. 3.3.31.

Now we will apply initial value theorem.

According to initial value theorem.

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$$x(0) = \lim_{z \to \infty} X(z)$$

$$\therefore \quad x(0) = \lim_{z \to \infty} \frac{7z}{z - \frac{1}{3}} - \frac{6z}{z - \frac{1}{2}}$$

Dividing numerator and denominator by z we get,

$$x(0) = \lim_{z \to \infty} \frac{7}{1 - \frac{1}{3z}} - \frac{6}{1 - \frac{1}{2z}} = 7 - 6 = 1$$

Now consider given equation,

$$x(n) = 7\left(\frac{1}{3}\right)^n u(n) - 6\left(\frac{1}{2}\right)^n u(n)$$

Putting n = 0 we get,

$$x(0) = 7\left(\frac{1}{3}\right)^0 u(0) - 6\left(\frac{1}{2}\right)^0 u(0) = 7 - 6 = 1$$

...us calculated z transfer is correct.

Thus calculated z transfer is correct.

3.4 Summary of the Properties of Z-transform :

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Table 3.4.1 shows the summary of Z-transform properties.

Table 3.4.1

	Property	Equation Time domain Z-domain	ROC
1.	Linearity	$a_1 x_1(n) + a_2 x_2(n) \overset{Z}{\longleftrightarrow}$ $a_1 X_1(Z) + a_2 X_2(Z)$	Intersection of the ROC of $X_1(Z)$ and $X_2(Z)$
2.	Time shifting	$x(n-k) \overset{Z}{\longleftrightarrow} Z^{-k} X(Z)$	Same as ROC of $X(Z)$ Except $Z = 0$ if $K > 0$ and $Z = \infty$ if $K < 0$
3.	Scaling in Z domain	$a^n x(n) \overset{Z}{\longleftrightarrow} X\left(\frac{Z}{a}\right)$	$\|a\| r_1 < \|Z\| < \|a\| r_2$
4.	Time reversal	$x(-n) \overset{Z}{\longleftrightarrow} X(Z^{-1})$	$\frac{1}{r_2} < \|Z\| < \frac{1}{r_1}$
5.	Differentiation in Z domain	$nx(n) \overset{Z}{\longleftrightarrow} -Z \frac{d}{dZ} X(Z)$	Same as ROC of $X(Z)$
6.	Convolution	$x_1(n) * x_2(n) \overset{Z}{\longleftrightarrow} X_1(2) \cdot X_2(Z)$	ROC is atleast intersection of ROC of $X_1(Z)$ and $X_2(Z)$

	Property	Equation Time domain Z-domain	ROC
7.	Correlation	$r_{x_1 x_2}(l) \overset{Z}{\longleftrightarrow} X_1(Z) \cdot X_2(Z^{-1})$	ROC is intersection of ROC of $X_1(Z)$ and $X_2(Z)$
8.	Multiplication	$X_1(n) \cdot x_2(n) \overset{Z}{\longleftrightarrow} \frac{1}{2\pi j} \oint_c X_1(v) X_2\left(\frac{Z}{v}\right) v^{-1} dv$	ROC is at least $r_{1l}\, r_{2l} \lvert Z \rvert < r_{1u} r_{2u}$
9.	Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n) x_*^2(n) \overset{Z}{\longleftrightarrow} \frac{1}{2\pi j} \oint_c X_1(v) X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv$	–
10.	Initial value theorem	Causal $x(n)$ $x(0) = \lim_{Z \to \infty} X(Z)$	–
	Final value theorem	Causal $x(n)$: $x(\infty) = \lim_{Z \to 1} X(Z)$	–

Ex. 3.3.30 : Determine the value of signal $x(n)$ at $n = 0$ and $n = \infty$ if,

$$X(Z) = \frac{2Z^2 + 0.25}{(Z + 0.25)(Z - 1)}$$

Soln. : According to initial value theorem,

$$x(0) = \lim_{Z \to \infty} X(Z)$$

$$x(0) = \lim_{Z \to \infty} \frac{2Z^2 + 0.25}{(Z + 0.25)(Z - 1)}$$

Multiplying numerator and denominator by Z^{-2});

$$x(0) = \lim_{Z \to \infty} \frac{2 + 0.25\,Z^{-2}}{Z^{-1}(Z + 0.25) \cdot Z^{-1}(Z - 1)} = \lim_{Z \to \infty} \frac{2 + \frac{0.25}{Z^2}}{\left(1 + \frac{0.25}{Z}\right)\left(1 - \frac{1}{Z}\right)}$$

$$\therefore \quad x(0) = \frac{2}{1} = 2$$

According to final value theorem,

$$x(\infty) = \lim_{Z \to 1} (Z - 1)\, X(Z)$$

$$\therefore \quad x(\infty) = \lim_{Z \to 1} (Z - 1) \cdot \frac{2Z^2 + 0.25}{(Z + 0.25)(Z - 1)} = \lim_{Z \to 1} \frac{2Z^2 + 0.25}{Z + 0.25} = \frac{2 + 0.25}{1 + 0.25}$$

$$\therefore x(\infty) = 1.8$$