LAPLACE DOMAIN ANALYSIS, SOLUTION OF DIFFERENTIAL EQUATIONS & SYSTEM BEHAVIOUR

Q.1 LTI System, which is initially at rest is described by differential equation, $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$

Calculate system transfer function and impulse response.

Solⁿ

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} \quad \text{—— (1)}$$

Taking Laplace transform on both sides,

$$s^2 Y(s) + 3s\,Y(s) + 2\,Y(s) = s\,X(s)$$

$$[s^2 + 3s + 2]\,Y(s) = s\,X(s)$$

Transfer function, $H(s) = \frac{Y(s)}{X(s)} = \frac{s}{s^2+3s+2}$ —— (2)

$$H(s) = \frac{s}{(s+2)(s+1)}$$

Using Partial fraction,

$$H(s) = \frac{A}{(s+1)} + \frac{B}{(s+2)} \quad \text{--- (3)}$$

$$A = (s+1)\frac{5}{(s+2)(s+1)}\Bigg|_{s=-1}$$

$$A = -1$$

$$B = (s+2)\frac{5}{(s+2)(s+1)}\Bigg|_{s=-2}$$

$$B = 2$$

Putting the value of A & B in eq^n (3)

$$H(s) = \frac{-1}{(s+1)} + \frac{2}{(s+2)}$$

Taking inverse Laplace transform, we get impulse response,

$$\boxed{h(t) = [-e^{-t} + 2e^{-2t}]\, u(t)}$$

Q.2

$$5\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 4y(t) = 3x(t)$$

for the system described by the above differential equation, determine whether the system is underdamped, overdamped or critically damped. Also find the impulse response of the system.

Solution - $5\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 4y(t) = 3x(t)$ —— ①

Taking Laplace transform of both sides of eq^n ①

$$5s^2Y(s) + 8sY(s) + 4Y(s) = 3X(s)$$

$$(5s^2 + 8s + 4)Y(s) = 3X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3}{5s^2 + 8s + 4}$$

$$H(s) = \frac{3/5}{s^2 + \frac{8}{5}s + \frac{4}{5}} \quad \text{—— ②}$$

comparing with $(s^2 + 2\xi\omega_n s + \omega_n^2)$

$$\omega_n^2 = \frac{4}{5} = 0.8$$

$$2\xi\omega_n = \frac{8}{5}$$

$$\boxed{\xi = 0.894}$$

As, $\xi < 1$, thus system is underdamped.

$$H(s) = \frac{3/5}{\left(s + \frac{4}{5} - \frac{2}{5}i\right)\left(s + \frac{4}{5} + \frac{2}{5}i\right)} \quad \text{—— ③}$$

Now we can solve eq^n ③ by partial fraction

$$H(s) = \frac{A}{\left(s + \frac{4}{5} - \frac{2}{5}i\right)} + \frac{B}{\left(s + \frac{4}{5} + \frac{2}{5}i\right)} \quad \text{—— ④}$$

After solving eq^n ④ we get

$$A = \frac{3}{4i}, \quad B = -\frac{3}{4i}$$

putting these values in eq^n ④

$$H(s) = \frac{3/4i}{\left(s+\frac{4}{5}-\frac{2}{5}i\right)} - \frac{3/4i}{\left(s+\frac{4}{5}+\frac{2}{5}i\right)}$$

$$H(s) = \frac{3}{4i}\left[\frac{1}{\left(s+\frac{4}{5}-\frac{2}{5}i\right)} - \frac{1}{\left(s+\frac{4}{5}+\frac{2}{5}i\right)}\right]$$

Taking inverse Laplace transform, the impulse response

$$h(t) = \frac{3}{4i}\left[e^{-\left(\frac{4}{5}-\frac{2}{5}i\right)t} - e^{-\left(\frac{4}{5}+\frac{2}{5}i\right)t}\right]u(t)$$

$$h(t) = \frac{3}{4i}\left[e^{-\frac{4}{5}t}\left(e^{\frac{2}{5}it} - e^{-\frac{2}{5}it}\right)\right]u(t)$$

$$h(t) = \frac{-3i}{4}e^{-\frac{4}{5}t}\left[2i\sin\left(\frac{2}{5}t\right)\right]u(t)$$

$$\boxed{h(t) = \frac{3}{2}e^{-\frac{4}{5}t}\sin\left(\frac{2}{5}t\right)u(t)}$$

Q.3 If $X(s) = \dfrac{2s+3}{(s+1)(s+2)}$, find $x(t)$ for

(i) System is stable
(ii) System is casual

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(iii) System is non-casual.

Solution- $X(s) = \frac{2s+3}{(s+1)(s+2)}$

Using partial fraction

$$X(s) = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

on solving partial fraction

$A = 1$, $B = 1$

$$X(s) = \frac{1}{(s+1)} + \frac{1}{(s+2)} \quad \text{--- } (1)$$

(i) If system is stable, its ROC should include $j\omega$ axis, thus ROC is $Re(s) > -1$

Now taking inverse Laplace transform of eqn (1)

$$x(t) = e^{-t}u(t) + e^{-2t}u(t)$$

(ii) If system is casual then ROC is $Re(s) > -1$, hence taking inverse Laplace transform of eqn (1)

$$x(t) = e^{-t}u(t) + e^{-2t}u(t)$$

(iii) If system is non-casual, then ROC is $Re(s) < -2$. Hence taking inverse laplace transform of eq^n (1)

$$x(t) = -e^{-t} u(-t) - e^{-2t} u(-t)$$

Q.4 Determine the total response of the differential eq^n

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

where $y(0) = 3$, $y'(0) = 4$, $x(t) = 4e^{-2t}$ and $t \geq 0$

Solution – $\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$

Taking Laplace transform

$$s^2 Y(s) - s y(0) - y'(0) + 3[s Y(s) - y(0)] + 2Y(s) = X(s)$$

$$Y(s)[s^2 + 3s + 2] - y(0) - y'(0) - 3y(0) = X(s)$$

$$Y(s)[s^2 + 3s + 2] - s \cdot 3 - 4 - 3 \times 3 = X(s)$$

$$Y(s)[s^2 + 3s + 2] - 3s - 13 = X(s)$$

$$Y(s)[s^2 + 3s + 2] - 3s - 13 = \frac{4}{s+2}$$

$$Y(s) = \frac{\frac{4}{s+2} + 3s + 13}{s^2 + 3s + 2}$$

$$Y(s) = \frac{4 + (3s+13)(s+2)}{(s+2)(s^2+3s+2)} = \frac{4 + 3s^2 + 19s + 26}{(s+2)(s^2+3s+2)}$$

$$Y(s) = \frac{3s^2 + 19s + 30}{(s+2)^2(s+1)}$$

$$Y(s) = \frac{3s^2 + 19s + 30}{(s+2)^2(s+1)}$$

Using partial fraction

$$Y(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+1)} \quad \text{--- (1)}$$

On solving partial fraction, we get

$$A = -11, \quad B = -4, \quad C = 14$$

$$Y(s) = \frac{-11}{s+2} + \frac{(-4)}{(s+2)^2} + \frac{14}{(s+1)}$$

$$y(t) = -11e^{-2t}u(t) - 4e^{-2t}u(t) + 14e^{-t}u(t)$$

$$\boxed{y(t) = -e^{-2t}(11 + 4t)u(t) + 14e^{-t}u(t)}$$

Q.5 Find the energy spectral ~~des~~ density of $f(t) = e^{-5t}u(t)$.

Soln Taking Fourier transform of

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

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$$X(\omega) = \int_{-\infty}^{\infty} e^{-5t} u(t) e^{-J\omega t} dt$$

$$= \int_{0}^{\infty} e^{-(5+J\omega)t} dt$$

$$= \frac{-1}{5+J\omega} \left[e^{-(5+J\omega)t} \right]_0^{\infty}$$

$$X(\omega) = \frac{1}{5+J\omega}$$

$$E \cdot D \cdot S = \left| \frac{1}{2J\alpha} \right|^2$$

$$= \frac{1}{4\alpha^2}$$

Ans