Fig. P. 3.7.8

3.7.3 Conditions for Causality and Stability in terms of Z-Transform :

We know that the condition for causality in terms of h(n) is,

$$h(n) = 0 \quad n < 0 \qquad ...(3.7.4)$$

Here the condition n < 0 indicates that value of h (n) for negative 'n' is zero. That h(n) is causal sequence. In Z-transforms we know that if a sequence is causal then the ROC is exterior part of circle having radius 'r'. Now the system transfer function is,

$$H(Z) = Z\{h(n)\} \qquad ...(3.7.5)$$

Thus the condition for causality is that the ROC of system transfer function should be exterior part of circle having radius 'r' and r < ∞.

We have already studied the condition for stability. It is,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \qquad ...(3.7.6)$$

Now $\mathcal{Z}\{h(n)\}$ can be written as,

$$\mathcal{Z}\{h(n)\} = H(Z) = \sum_{n=-\infty}^{\infty} h(n) Z^{-n}$$

Taking absolute value on both the sides,

$$|H(Z)| = \left| \sum_{n=-\infty}^{\infty} h(n) Z^{-n} \right| \quad ...(3.7.7)$$

Now the absolute value on a sum is always less than the sum of absolute value.

$$\therefore |H(Z)| < \sum_{n=-\infty}^{\infty} |h(n)| |Z^{-n}| \quad ...(3.7.8)$$

If we evaluate the Z-transform on the unit circle that means $|Z^{-n}| = |Z| = 1$ then Equation (3.7.9) becomes,

$$|H(Z)| < \sum_{n=-\infty}^{\infty} |h(n)| \quad ...(3.7.10)$$

According to Equation (3.7.6), if the system is BIBO stable then $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$. Thus the condition of stability is Z-domain is,

$$|H(Z)| < \infty \text{ when evaluated on unit circle.} \quad ...(3.7.11)$$

That means the system transfer function should be finite if evaluated on unit circle.

Equation (3.7.6) gives the condition of stability in Z domain. ***This condition requires that, unit circle must be present in the ROC of H(Z)***. Otherwise we cannot find H(Z) on unit circle at all.

Combined Condition :

(1) For a causal LSI system we know that ROC is exterior part of circle; that mean $|Z| > |r|$

(2) For the stable LSI system, ROC of H(Z) must include unit circle. Here unit circle is a circle having radius 1. Now consider we have a causal system having ROC as exterior part of circle with radius 'r'. Then for this system to be stable; it should include unit circle as shown in Fig. 3.7.1.

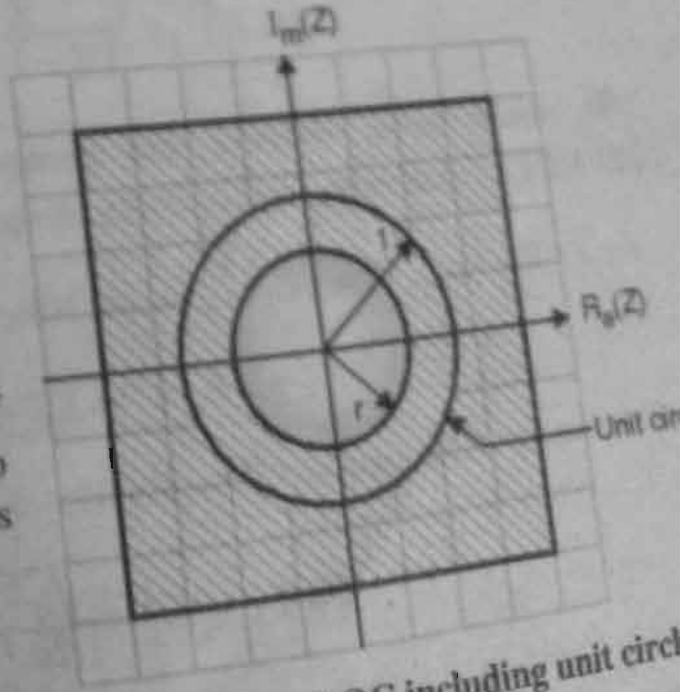

Fig. 3.7.1 : ROC including unit circl

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Thus the combined condition for causal and stable LSI system is,

$$\boxed{|Z| > r < 1}$$

We know that,

(i) *ROC of H(Z)* does not contain any poles.

(ii) *ROC of stable* and causal LSI system includes the unit circle.

Thus we can state that ***all the poles of H(Z) of a causal and stable system a[...]***
circle.

Ex. 3.7.9 : Consider the system $H(Z) = \dfrac{1-2Z^{-1}+2Z^{-2}-Z^{-3}}{(1-Z^{-1})(1-0.5Z^{-1})(1-0.2Z^{-1})}$

with ROC : $0.5 < |Z| < 1$.

(i) Sketch the pole-zero pattern. Is the system stable ?

(ii) Determine the impulse response of system.

Soln. :

(i) **Pole-zero Pattern :** The given function is,

$$H(Z) = \frac{1-2Z^{-1}+2Z^{-2}-Z^{-3}}{(1-Z^{-1})(1-0.5Z^{-1})(1-0.2Z^{-1})}$$

To convert it into positive powers of Z we will multiply numerator and denominator by Z^3.

$$\therefore\ H(Z) = \frac{Z^3(1-2Z^{-1}+2Z^{-2}-Z^{-3})}{Z(1-Z^{-1})Z(1-0.5Z^{-1})Z(1-0.2Z^{-1})}$$

$$\therefore\ H(Z) = \frac{Z^3-2Z^2+2Z-1}{(Z-1)(Z-0.5)(Z-0.2)}$$

To obtain zeros, consider the numerator polynomial. We will obtain the roots Z^3-2Z^2+2Z-1. Let us perform synthetic division.

$1-2+2-1=0$. Thus $(Z-1)$ exists. Now dividing Z^3-2Z^2+2Z-1 by $(Z-1)$ we get,

$$Z^3-2Z^2+2Z-1 = (Z-1)(Z^2-Z+1)$$

Now we will obtain roots of Z^2-Z+1

$$Z^2-Z+1 = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{1\pm\sqrt{1-4}}{2} = \frac{1}{2}\pm j\frac{\sqrt{3}}{2}$$

$$\therefore\ Z^2-Z+1 = \left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)$$

Thus Equation (1) becomes,

$$Z^3 - 2Z^2 + 2Z - 1 = (Z-1)\left(Z - \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\left(Z - \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) \quad ...(4)$$

Putting Equation (4) in Equation (2) we get,

$$H(Z) = \frac{(Z-1)\left(Z - \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\left(Z - \frac{1}{2} + j\frac{\sqrt{3}}{2}\right)}{(Z-1)(Z-0.5)(Z-0.2)}$$

$$\therefore H(Z) = \frac{\left(Z - \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\left(Z - \frac{1}{2} + j\frac{\sqrt{3}}{2}\right)}{(Z-0.5)(Z-0.2)} \quad ...(5)$$

We have the standard equation of H (Z) in terms of poles and zeros,

$$H(Z) = \frac{(Z-Z_1)(Z-Z_2)}{(Z-P_1)(Z-P_2)} \quad ...(6)$$

Comparing Equations (5) and (6) we can write,

Zeros $\Rightarrow Z_1 = \frac{1}{2} + j\frac{\sqrt{3}}{2}$ and $Z_2 = \frac{1}{2} - j\frac{\sqrt{3}}{2}$

and Poles $\Rightarrow P_1 = 0.5$ and $P_2 = 0.2$.

The pole-zero plot is shown in Fig. P. 3.7.9.

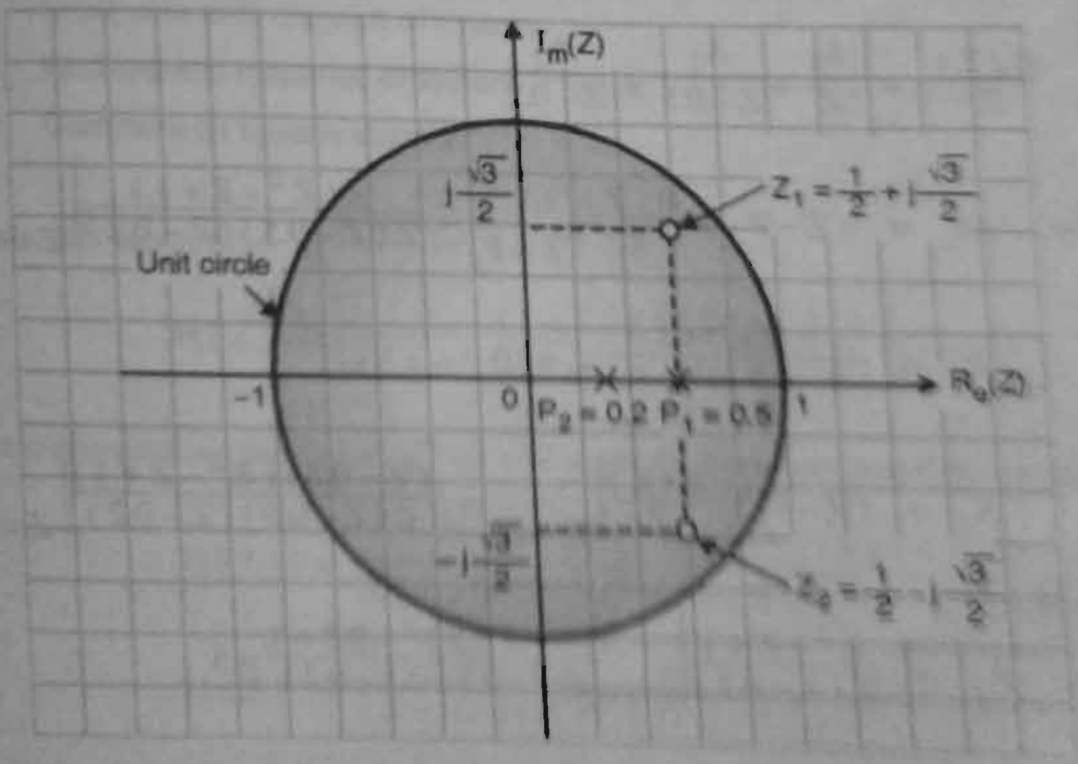

Fig. P. 3.7.9 : Pole-zero plot

(ii) Impulse Response :

Impulse response is denoted by h (n). It is obtained by taking IZT of H (Z).

We have,

$$H(Z) = \frac{\left(Z-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\left(Z-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)}{(Z-0.5)(Z-0.2)}$$

$$\therefore \quad H(Z) = \frac{Z^2-Z+1}{Z^2-0.7Z+0.1}$$

Here M = N; we want M < N. Thus multiplying and dividing by Z,

$$H(Z) = \frac{Z(Z^2-Z+1)}{Z(Z^2-0.7Z+0.1)}$$

$$\therefore \quad \frac{H(Z)}{Z} = \frac{Z^2-Z+1}{Z(Z^2-0.7Z+0.1)}$$

Now the degree of numerator (M) is less than the degree of denominator (N). So it is in proper form.

$$\therefore \frac{H(Z)}{Z} = \frac{Z^2-Z+1}{Z(Z-0.5)(Z-0.2)}$$

Here poles are $P_1 = 0$, $P_2 = 0.5$ and $P_3 = 0.2$.

We will write Equation (9) in partial fraction expansion form.

$$\therefore \quad \frac{H(Z)}{Z} = \frac{A_1}{Z} + \frac{A_2}{Z-0.5} + \frac{A_3}{Z-0.2}$$

Now $A_1 = (Z-P_1)\left.\frac{H(Z)}{Z}\right|_{Z=P_1} = \left.\frac{Z(Z^2-Z+1)}{Z(Z-0.5)(Z-0.2)}\right|_{Z=0}$

$$\therefore \quad A_1 = \frac{0+0+1}{(0-0.5)(0-0.2)} = 10$$

$$A_2 = (Z-P_2)\left.\frac{H(Z)}{Z}\right|_{Z=P_2} = \left.\frac{(Z-0.5)(Z^2-Z+1)}{Z(Z-0.5)(Z-0.2)}\right|_{Z=0.5}$$

$$\therefore \quad A_2 = \left.\frac{Z^2-Z+1}{Z(Z-0.2)}\right|_{Z=0.5} = \frac{(0.5)^2-0.5+1}{0.5(0.5-0.2)} = 5$$

and $A_3 = (Z-P_3)\left.\frac{H(Z)}{Z}\right|_{Z=P_3} = \left.\frac{(Z-0.2)(Z^2-Z+1)}{Z(Z-0.5)(Z-0.2)}\right|_{Z=0.2}$

$$\therefore \quad A_3 = \left.\frac{Z^2-Z+1}{Z(Z-0.5)}\right|_{Z=0.2} = \frac{(0.2)^2-0.2+1}{0.2(0.2-0.5)} = -14$$

...ting these values in Equation (10) we get,

$$\frac{H(Z)}{Z} = \frac{10}{Z} + \frac{5}{Z-0.5} + \frac{(-14)}{Z-0.2}$$

$$\therefore H(Z) = 10 + \frac{5Z}{Z-0.5} - \frac{14Z}{Z-0.2} \quad \text{...(1}$$

The given ROC is $0.5 < |Z| < 1$. So all the terms are causal.

$$\therefore IZT\{10\} = 10\, IZT\{1\} = 10\,\delta(n)$$

$$IZT\left\{\frac{5Z}{Z-0.5}\right\} = 5\, IZT\left\{\frac{Z}{Z-0.5}\right\} = 5(0.5)^n u(n)$$

and

$$IZT\left\{\frac{-14Z}{Z-0.2}\right\} = -14\, IZT\left\{\frac{Z}{Z-0.2}\right\} = -14(0.2)^n u(n).$$

$$\therefore h(n) = 10\,\delta(n) + 5(0.5)^n u(n) - 14(0.2)^n u(n)$$