6.14. Laplace Transforms of Few Important Functions

In this section, let us evalute the Laplace transform of some imnportant functions.

6.14.1. Unit Step Function

Here, $$x(t) = u(t) = \begin{cases} 1, \text{ for } 0 < t < \infty \\ 0, \text{ for } t \leq 0 \end{cases}$$

Therefore $$£\{u(t)\} = \int_0^\infty 1.e^{-st} dt = -\frac{1}{s}[e^{-st}]_0^\infty = -\frac{1}{s}[0-1] = \frac{1}{s}$$

Thus $$£[u(t)] = \frac{1}{s}$$

6.14.2. Exponential function

Here, $$x(t) = Ae^{-at}$$

Therefore, $$£\{Ae^{-at}\} = \int_0^\infty Ae^{-at}e^{-st} dt = A\int_0^\infty e^{-(a+s)t} dt$$

$$= -\frac{A}{a+s}[e^{-(a+s)t}]_0^\infty = \frac{A}{(s+a)}$$

Thus, $$£\{Ae^{-at}\} = \frac{A}{(s+a)}$$

6.14.3. Sine Function

Here, $x(t) = \sin \omega_0 t$

Using Euler's identity, we get

$$\sin \omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$$

Therefore, $\pounds\{\sin \omega_0 t\} = \frac{1}{2j}\left[\pounds(e^{j\omega_0 t}) - \pounds(e^{-j\omega_0 t})\right]$

Simplifhying, we get

$$\pounds[\sin \omega_0 t] = \frac{1}{2j}\left[\frac{1}{s - j\omega_0} - \frac{1}{s + j\omega_0}\right] = \frac{\omega_0}{s^2 + \omega_0^2}$$

Hence, $$\pounds\{\sin \omega_0 t\} = \frac{\omega_0}{s^2 + \omega_0^2}$$

6.14.4. Cosine Function

Here, $x(t) = \cos \omega_0 t$

We know that

$$\cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

Thus, $$\pounds\{\cos \omega_0 t\} = \frac{1}{2}[\pounds(e^{j\omega_0 t} + \pounds(e^{-j\omega_0 t})]$$

$$= \frac{1}{2}\left[\frac{1}{s - j\omega_0} + \frac{1}{s + j\omega_0}\right] = \frac{s}{s^2 + \omega_0^2}$$

Hence, $$\pounds\{\cos \omega_0 t\} = \frac{s}{s^2 + \omega_0^2}$$

6.14.5. Hyperbolic Sine and Cosine Functions

Here $$\sin h\, \omega_0 t = \frac{1}{2}[e^{\omega_0 t} - e^{-\omega_0 t}]$$

and $$\cos h\, \omega_0 t = \frac{1}{2}[e^{\omega_0 t} + e^{-\omega_0 t}]$$

Hence, $$\pounds\{\sin h\, \omega_0 t) = \frac{1}{2}[\pounds(e^{\omega_0 t}) - \pounds(e^{-\omega_0 t})]$$

Simplifying, we get

$$\pounds(\sin \omega_0 t) = \frac{1}{2}\left[\frac{1}{s - \omega_0} - \frac{1}{s + \omega_0}\right] = \frac{\omega_0}{s^2 - \omega_0^2}$$

Therefore, $$\pounds\{\sin h\, \omega_0 t\} = \frac{\omega_0}{s^2 - \omega_0^2}$$

Similarly we can write,

$$\pounds\{\cosh \omega_0 t\} = \frac{1}{2}[\pounds(e^{\omega_0 t}) + \pounds(e^{-\omega_0 t})]$$

Simplifying, we get

$$\pounds\{\cos h\, \omega_0 t) = \frac{1}{2}\left[\frac{1}{s - \omega_0} + \frac{1}{s + \omega_0}\right] = \frac{s}{s^2 - \omega_0^2}$$

Hence, $$\pounds\{\cos h\, \omega_0 t\} = \frac{s}{s^2 - \omega_0^2}$$

6.14.6. Damped Sine and Cosine Functions

Here, we have

$$\mathcal{L}[e^{-at}\sin\omega_0 t] = \mathcal{L}\left[e^{-at}\left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right)\right]$$

$$= \frac{1}{2j}\left[\mathcal{L}\left\{e^{-(a-j\omega_0)t} - e^{-(a+j\omega_0)t}\right\}\right]$$

Simplifying, we get

$$\mathcal{L}[e^{-at}\sin\omega_0 t] = \frac{1}{2j}\left[\frac{1}{s+(a-j\omega_0)} - \frac{1}{s+(a+j\omega_0)}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{(s+a)-j\omega_0} - \frac{1}{(s+a)+j\omega_0}\right]$$

or $$\mathcal{L}[e^{-at}\sin\omega_0 t] = \frac{\omega_0}{(s+a)^2+\omega_0^2}$$

Hence, $$\mathcal{L}[e^{-at}\sin\omega_0 t] = \frac{\omega_0}{(s+a)^2+\omega_0^2}$$

Similarly, $$\mathcal{L}[e^{-at}\cos\omega_0 t] = \mathcal{L}\left\{e^{-at}\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right]\right\}$$

Simplifying, we get

$$\mathcal{L}[e^{-at}\cos\omega_0 t] = \frac{s+a}{(s+a)^2+\omega_0^2}$$

Hence, $$\mathcal{L}[e^{-at}\cos\omega_0 t] = \frac{s+a}{(s+a)^2+\omega_0^2}$$

6.14.7. Damped Hyperbolic sine and cosine Functions

Here, we have

$$\mathcal{L}\{e^{-at}\sinh\omega_0 t\} = \mathcal{L}\left\{e^{-at}\left(\frac{e^{\omega_0 t} - e^{-\omega_0 t}}{2}\right)\right\}$$

$$= \frac{1}{2}\mathcal{L}(e^{-(a-\omega_0)t}) - \mathcal{L}(e^{-(a+\omega_0)t})]$$

Simplifying, we get

$$\mathcal{L}\{e^{-at}\sin h\,\omega_0 t\} = \frac{1}{2}\left[\frac{1}{s+a-\omega_0} - \frac{1}{s+a+\omega_0}\right] = \frac{\omega_0}{(s+a)^2-\omega_0^2}$$

Hence, $$\mathcal{L}\{e^{-at}\sinh\omega_0 t\} = \frac{\omega_0}{(s+a)^2-\omega_0^2}$$

Similarly, $$\mathcal{L}\{e^{-at}\cosh\omega_0 t) = \frac{s+a}{(s+a)^2-\omega_0^2}$$

6.14.8. t^n Function

Here, we have

$$\pounds\{t^n\} = \int_0^\infty t^n e^{-st} dt = \int_0^\infty t^n d\left(\frac{e^{-st}}{-s}\right)$$

Simplifying, we get

$$\pounds\{t^n\} = \left[\frac{t^n e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} n t^{n-1} dt$$

$$\pounds\{t^n\} = \frac{n}{s}\int_0^\infty e^{-st}\, t^{n-1}\, dt = \frac{n}{s}\pounds(t^{n-1})$$

Similarly, $\pounds\{t^{n-1}\} = \frac{n-1}{s}\pounds(t^{n-2})$

Now, taking Laplace transformations of t^{n-2}, t^{n-3}, ... and substituting in the above equation, we get

$$\pounds\{t^n\} = \frac{n}{s}\frac{n-1}{s}\frac{n-2}{s}\cdots\frac{2}{s}\frac{1}{s}\pounds(t^{n-n})$$

$$= \frac{n!}{s^n}\pounds(t^0) = \frac{n!}{s^n} \times \frac{1}{s} = \frac{n!}{s^{n+1}},$$

when n is positive integer.

Hence, $\pounds\{t^n\} = \frac{n!}{s^{n+1}}$

Substituting $n = 1$, we have $\pounds\{t\} = 1/s^2$.

Table 6.1. Some Laplace Transform pairs

S.No.	$x(t)$	$X(s)$	ROC
1.	$\delta(t)$	1	All s
2.	$u(t)$	$1/s$	$Re(s) > 0$
3.	$-u(-t)$	$1/s$	$Re(s) < 0$
4.	$tu(t)$	$\frac{1}{s^2}$	$Re(s) > 0$
5.	$t^k u(t)$	$\frac{k!}{s^{k+1}}$	$Re(s) > 0$
6.	$e^{-at}u(t)$	$\frac{1}{s+a}$	$Re(s) > -Re(a)$
7.	$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$Re(s) < -Re(a)$
8.	$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$Re(s) > -Re(a)$
9.	$-te^{-at}u(-t)$	$\frac{1}{(s+a)^2}$	$Re(s) < -Re(a)$

S.No.	$x(t)$	$X(s)$	ROC
10.	$\cos \omega_0 tu(t)$	$\frac{s}{s^2+\omega_0^2}$	$Re(s) > 0$
11.	$\sin \omega_0 tu(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$Re(s) > 0$
12.	$e^{-at} \cos \omega_0 tu(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$Re(s) > -Re(a)$
13.	$e^{-at} \sin \omega_0 tu(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$Re(s) > -Re(a)$

Example 6.27. Determine the Laplce transform of the following given functions

(*i*) $x(t) = t^3 + 3t^2 - 6t + 4$ (*ii*) $x(t) = \cos^3 3t$

(*iii*) $x(t) = \sin at \cos bt$ (*iv*) $x(t) = t \sin at$

(*v*) $x(t) = \frac{1-e^t}{t}$ (*vi*) $x(t) = \delta(t^2 - 3t + 2)$

Solution : (*i*) Given function is

$$x(t) = t^3 + 3t^2 - 6t + 4$$

Therefore $x(t) = £[t^3 + 3t^2 - 6t + 4]$

Simplifying, we get

$$£[x(t)] = \frac{3!}{s^4} + 3\frac{2!}{s^3} - 6\frac{1!}{s^2} + \frac{4}{s} = \frac{6}{s^4} + \frac{6}{s^3} - \frac{6}{s^2} + \frac{4}{s} \quad \text{Ans.}$$

(*ii*) Given function is

$$x(t) = \cos^3 3t \qquad ...(i)$$

We know thatcos $3A = 4\cos^3 A - 3\cos A$

Therefore, equation (*i*) can be written as

$$x(t) = \cos^3 3\text{t} = \frac{1}{4}[\cos 9t + 3\cos 3t]$$

Thus, we have,

$$£\{\cos^3 3t\} = £\left[\frac{\cos 9t + 3\cos 3t}{4}\right] = \frac{1}{4}\left[\frac{s}{s^2+9^2} + 3\frac{s}{s^2+3^2}\right]$$

or $$£[\cos^3 3t] = \frac{1}{4}\left[\frac{s}{s^2+81} + \frac{3s}{s^2+9}\right] \quad \text{Ans.}$$

(*iii*) Given function is

$$x(t) = \sin at \cos bt$$

or $$x(t) = \frac{1}{2}\{\sin(9+b)t + \sin(a-b)t\}$$

$$£\{\sin at \cos bt\} = £\left[\frac{1}{2}\{\sin(a+b)t + \sin(a-b)t\}\right]$$

Simiplifying, we get

$$£\{\sin at \cos bt\} = \frac{1}{2}\left[\frac{a+b}{s^2+(a+b)^2} + \frac{a-b}{s^2+(a-b)^2}\right] \quad \text{Ans.}$$

Given function is

$$x(t) = t.\sin at$$

Therefore

(iv) $$£\{t \sin at\} = -\frac{d}{ds}£\{\sin at\}$$

$$£\{t \sin at\} = -\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right] = -a\frac{d}{ds}[(s^2+a^2)^{-1}]$$

or $$£\{t \sin at\} = -a\left[-\frac{1}{(s^2+a^2)^2}.2s\right] = \frac{2s}{(s^2+a^2)^2}$$ **Ans.**

Given function is

(v) $$x(t) = \left[\frac{1-e^t}{t}\right]$$

Here, since $£\{1-e^t\} = \frac{1}{s} - \frac{1}{(s-1)}$

Therefore, $$£\left\{\frac{1-e^t}{t}\right\} = \int_s^\infty \left[\frac{1}{s} - \frac{1}{(s-1)}\right] ds = [\log s - \log(s-1)]_s^\infty$$

or $$£\left\{\frac{1-e^t}{t}\right\} = \left[\log\frac{s}{s-1}\right]_s^\infty = \left[-\log\frac{s-1}{s}\right]_s^\infty$$

$$= -\log\left(1-\frac{1}{s}\right)\Big]_s^\infty = -\log\left(1-\frac{1}{s}\right)$$

or $$£\left\{\frac{1-e^t}{t}\right\} = \log\left(\frac{s-1}{s}\right)$$ **Ans.**

(vi) The given impulse function is

$$x(t) = \delta(t^2 - 3t + 2) = \delta[(t-1)(t-2)]$$

or $$x(t) = \delta(t-1)\,u(t-1) + \delta(t-2)\,u(t-2)$$

or $$x(t) = \delta(t-1) + \delta(t-2)$$

Therefore, $$X(s) = £[\delta(t-1)] + £[\delta(t-2)] = e^{-s} + e^{-2s}$$ **Ans.**

Example 6.28. Find the Laplace transform of the rectangular pulse shown in Figure 6.35.

Solution : W1know that Laplace Transform is given by

$$X(s) = £[x(t)] = \int_0^\infty x(t)\, e^{-st}\, dt$$

Thus, according to figure, we have

$$X(s) = \int_0^T 1 \cdot e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^T$$

$$= \frac{1}{-s}[e^{-sT} - 1] = \frac{1}{s}[1 - e^{-sT}]$$ **Ans.**

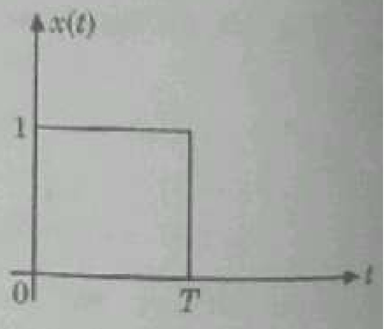

Fig. 6.35. A rectangular pulse.

Example 6.29. Find the Laplace transform of a single sawtooth pulse shown in Figure 6.36.

Solution : We can write the function for the given wave form as under:

$$x(t) = \begin{cases} t & \text{for } 0 < t < 1 \\ 0 & \text{for } t > 1 \end{cases} \quad ...(i)$$

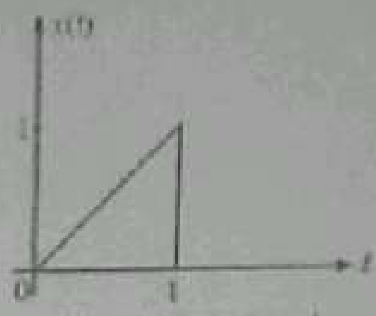

Fig. 6.36. A signal sawtooth pulse

We know that Laplace transform is given by

$$X(s) = \pounds[x(t)] = \int_0^\infty x(t)e^{-st}\,dt$$

Using equation (i), we get

or $$X(s) = \int_0^1 te^{-st}\,dt = \left[t\frac{e^{-st}}{-s} - 1\frac{e^{-st}}{s^2}\right]_0^1$$

or $$X(s) = \left[\left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right) - \left(0 - \frac{1}{s^2}\right)\right] = \frac{1}{s^2} - e^{-s}\left[\frac{1}{s} + \frac{1}{s^2}\right] \quad \text{Ans.}$$

Example 6.30. Determine the Laplace transform of the triangular pulse shown in Figure 6.37. *(A.M.I.E. Examination, 1987)*

Solution : For the given triangular waveform we write

$$x(t) = \begin{cases} \frac{2}{T}t, & \text{for } 0 \le t \le T/2 \\ 2 - \frac{2}{T}t, & \text{for } T/2 \le t \le T \end{cases} \quad ...(i)$$

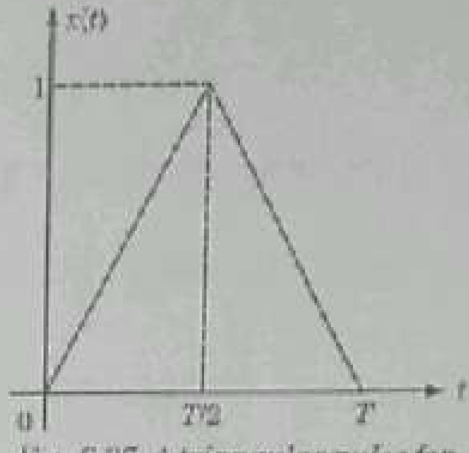

Fig. 6.37. A triangular pulse for example 6.30

We know that the Laplace transform is given by

Thus, $$X(s) = \pounds[x(t)] = \int_0^\infty x(t)e^{-st}\,dt$$

Using equation (i), we get

$$X(s) = \int_0^{T/2}\left(\frac{2}{T}t\right)e^{-st}dt + \int_{T/2}^{T}\left(2 - \frac{2}{T}t\right)e^{-st}dt$$

$$X(s) = \frac{2}{T}\int_0^{T/2} te^{-st}dt + 2\int_{T/2}^{T} e^{-st}dt - \frac{2}{T}\int_{T/2}^{T} te^{-st}dt$$

Simplifying, we get

$$X(s) = \frac{2}{T}\left[\left\{t\frac{e^{-st}}{-s}\right\} - \left\{\frac{e^{-st}}{s^2}\right\}\right]_0^{T/2}$$

$$= +2\left[\frac{e^{-st}}{-s}\right]_{T/2}^{T} - \frac{2}{T}\left[\left\{t\frac{e^{-st}}{-s}\right\} - \left\{\frac{e^{-st}}{s^2}\right\}\right]_{T/2}^{T}$$

or $$X(s) = \frac{2}{T}\left[\frac{T}{2}\frac{e^{-sT}}{-s} - 0 - \frac{e^{-sT/2}}{s^2} + \frac{1}{s^2}\right] - \frac{2}{s}[e^{-sT} - e^{-sT/2}]$$

$$-\frac{2}{T}\left[T\frac{e^{-sT}}{-s} - \frac{T}{2}\frac{e^{-sT/2}}{s^1} - \frac{e^{-sT}}{s^2} - \frac{e^{-sT/2}}{s^2}\right]$$

or $$X(s) = -\frac{e^{-sT/2}}{s} - \frac{2}{T}\frac{e^{-sT/2}}{s^2} + \frac{2}{T}\frac{1}{s^2} - \frac{2}{s}e^{-sT} + \frac{2}{s}e^{-sT/2}$$

$$+2\frac{e^{-sT}}{s} - \frac{e^{-sT/2}}{s} + \frac{2}{T}\frac{e^{-sT}}{s^2} - \frac{2}{T}\frac{e^{-sT/2}}{s^2}$$

or $$X(s) = \frac{2}{T}\frac{1}{s^2} - \frac{4}{T}\frac{e^{-sT/2}}{s^2} + \frac{2}{T}\frac{e^{-sT}}{s^2} \quad \textbf{Ans.}$$

Example 6.31. Determine the Laplace transform of the square wave shown in figure 6.38.

Solution : For the given wave form, we write

$$x(t) = \begin{cases} 1, \; for \; 0 < t < T \\ -1, \; for \; T < t 2T \end{cases} \quad ...(i)$$

We know that the Laplace tranform is given as

$$X(s) = \text{£}\, x\{(t)\} = \int_0^{\infty} x(t)\, e^{-st} dt$$

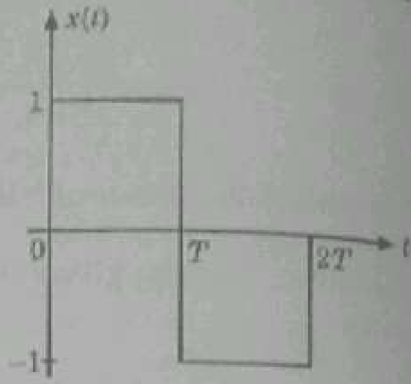

Fig. 6.38. Square waveform

Now, using equation (i), we get

or $$X(s) = \int_0^T 1 \cdot e^{-st} dt + \int_T^{2T} (-1) e^{-st} dt = \left[\frac{e^{-st}}{-s}\right]_0^T - \left[\frac{e^{-st}}{-s}\right]_T^{2T}$$

or $$X(s) = -\frac{1}{s}[e^{-sT} - 1 - e^{-2sT} + e^{-sT}]$$

$$= \frac{1}{s}[1 - 2e^{-sT} + e^{-2sT}] = \frac{1}{s}(1 - e^{-sT})^2 \quad \textbf{Ans.}$$

Example 6.32. For the waveform shown in figure 6.39, evaluate the Laplace transform.

Solution : The function for the given waveform will

$$x(t) = \begin{cases} A \sin t \; for \; 0 < t < \pi \\ 0, \quad t > \pi \end{cases} \quad ...(i)$$

We know that Laplace transform is given as

$$X(s) = \text{£}[x(t)] = \int_0^{\infty} x(t)\, e^{-st}\, dt$$

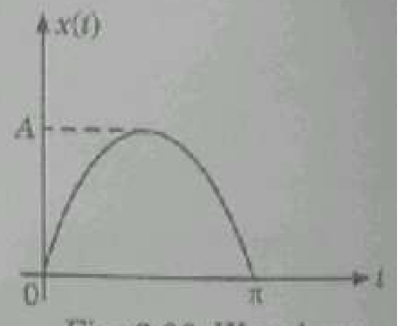

Fig. 6.39. Waveform for example 6.32.

Using equation (i), we get

$$X(s) = \int_0^{\pi} A \sin t\, e^{-st}\, dt = A\int_0^{\pi} \sin t\, e^{-st}\, dt$$

or $$X(s) = \frac{A}{(s^2+1)}[e^{-st} - s\sin t - \cos t]_0^{\pi}$$

or $$X(s) = A\frac{e^{-s\pi}+1}{(s^2+1)} \quad \text{Ans.}$$

6.15. Properties of the Laplace Transform

Now, let us discuss some bassic properties of the Laplace transfrm in this section.

6.15.1. Linearity

If $\quad x_1(t) \leftrightarrow X_1(s) \qquad \text{ROC} = R_1$

$\qquad x_2(t) \leftrightarrow X_2(s) \qquad \text{ROC} = R_2$

Then $\quad a_1x_1(t) + a_2x_2(t) \leftrightarrow a_1X_1(s) + a_2X_2(s) \qquad R' \supset R_1 \cap R_2$ ...(6.55)

The set notation $A \supset B$ means that set A contains set B, while $A \cap B$ denotes the intersection of sets A and B, i.e., the set containing all elements in both A and B. Thus, equation (6.55) shows that the ROC of the resultant Laplace transform is at least as large as the region in common between R_1 and R_2. Usually, we have simply $R' = R_1 \cap R_2$. Thus has been illustrated in figure 6.40.

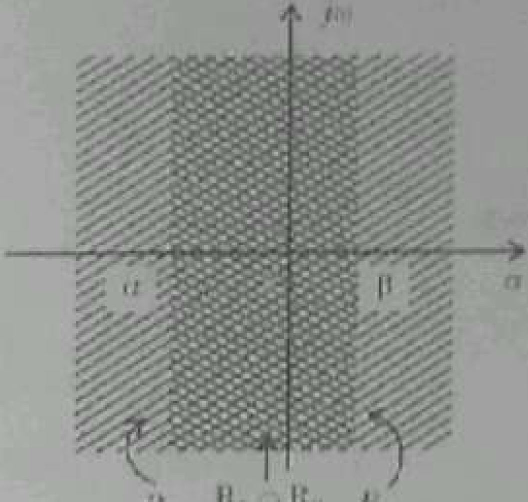

Fig. 6.40. ROC of $a_1 X_1(s) + a_2 X_2(s)$

6.15.2. Time Shifting

If

$\qquad x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$

then $\quad x(t - t_0) \leftrightarrow e^{-st_0}X(s) \qquad R' = R$ ...(6.56)

Equation (6.56) implies that the ROCs before and after the time-shift operation are the same.

6.15.3. Shifting in the s-Domain

If $\qquad x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$

then $\qquad e^{s_0t}\, x(t) \leftrightarrow X(s - s_0) \qquad R' = R + Re(s_0)$ ...(6.57)

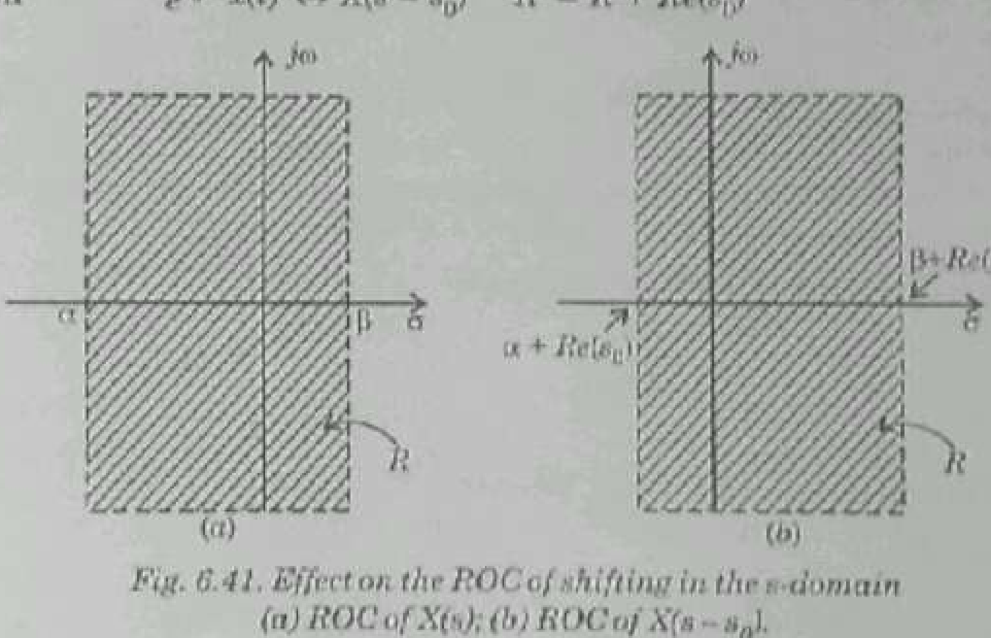

Fig. 6.41. Effect on the ROC of shifting in the s-domain (a) ROC of $X(s)$; (b) ROC of $X(s - s_0)$.

Equation (6.57) implies that the ROC associated with $X(s - s_0)$ is that of $X(s)$ shifted by $Re(s_0)$. This has been illustrated in figure 6.41.

6.15.4. Time Scaling

If $\quad x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$

then $\quad x(at) \leftrightarrow \dfrac{1}{|a|} X\left(\dfrac{s}{a}\right) \qquad R' = aR \qquad ...(6.58)$

Equation (6.58) shows that scaling the time variable t by the factor a causes an inverse scaling of the variable s by $1/a$ as well as an amplitude scaling of $X(s/a)$ by $1/|a|$. The corresponding effect on the ROC is illustrated in figure 6.42.

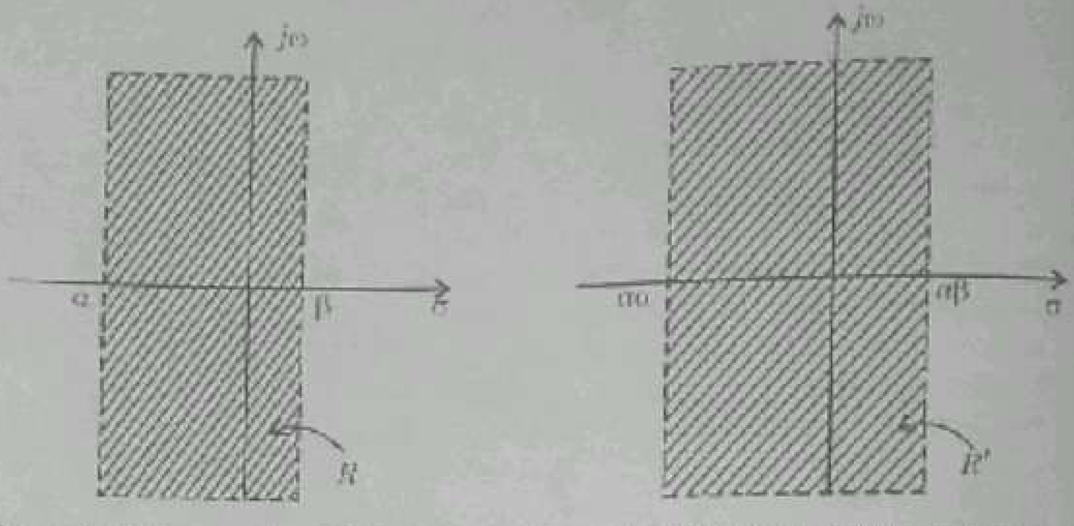

Fig. 6.42. Effect on the ROC of time scaling (a) ROC of X(s); (b) ROC of X(s/a).

6.15.5. Time Reversal

$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$

then $\quad x(-t) \leftrightarrow X(-s) \qquad R' = -R \qquad ...(6.59)$

Thus, time reversal of $x(t)$ produces a reversal of both the σ-and $j\omega$-axes in the s-plane. Equation (6.59) is readily obtained by setting $a = -1$ in equation (6.58).

6.15.6. Differentiation in the Time Domain

If $\quad x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$

then $\quad \dfrac{dx(t)}{dt} \leftrightarrow sX(s) \qquad R' \supset R \qquad ...(6.60)$

Equation (6.60) shows that the effect of differentiation in time domain is multiplication of the corresponding Laplace transform by s. The associated ROC is unchanged unless there is a pole-zero cancellation at $s = 0$.

6.15.7. Differentiation in the s-Domain

If $\quad x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$

then $\quad -tx(t) \leftrightarrow \dfrac{dX(s)}{ds} \qquad R' = R \qquad ...(6.61)$

6.15.8. Integration in the Time Domain

If $\quad x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$

then $\quad \displaystyle\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{1}{s} X(s) \qquad R' = R \cap \{Re(s) > 0\} \qquad ...(6.62)$

Equation (6.62) shows that the Laplace transform operation coresponding to time-domain integation is multiplication by 1/s, and this is expected since integration is the inverse operation of diffentiation. The form of R′ follows from the possible introduction of an additional pole at s = 0 by the multiplication by 1/s.

6.15.9. Convolution

If

$$x_1(t) \leftrightarrow X_1(s) \qquad \text{ROC} = R_1$$
$$x_2(t) \leftrightarrow X_2(s) \qquad \text{ROC} = R_2$$

then

$$x_1(t)\, x_2(t) \leftrightarrow X_1(s)\, X_2(s) \qquad R' \supset R_1 \cap R_2 \qquad \text{...(6.64)}$$

Table 6.2. Properties of the laplace transform

S.No.	Property	Signal	Transform	ROC
1.		$x(t)$	$X(s)$	R
2.		$x_1(t)$	$X_1(s)$	R_1
3.		$x_2(t)$	$X_2(s)$	R_2
4.	Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$	$R' \supset R_1 \cap R_2$
5.	Time shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	$R' = R$
6.	Shifting in s	$e^{s_0t}\,x(t)$	$X(s-s_0)$	$R' = R + Re(s_0)$
7.	Time scaling	$x(at)$	$\frac{1}{\|a\|}X(s)$	$R' = aR$
8.	Tme reversal	$x(-t)$	$X(-s)$	$R' = -R$
9.	Differentiation in t	$\frac{dx(t)}{dt}$	$sX(s)$	$R' \supset R$
10.	Differentiation in s	$-tx(t)$	$\frac{dX(s)}{ds}$	$R' = R$
11.	Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	$R' \supset R \cap \{(Re(s) > 0)\}$
12.	Convolution	$x_1(t)\,x_2(t)$	$X_1(s)\,X_2(s)$	$R' \supset R_1 \cap R_2$

This convolution property plays a central role in the analysis and design of continuous-time LTI systems.

Table 6.2. summarizes the properties of the Laplace transform presented in this section.