A.K.T.U. 2018

Moment Generating Function of Binomial Distribution

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1. About Origin: $M_x(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx}\, {}^nC_x\, p^x q^{n-x}$

$$= \sum_{x=0}^{n} {}^nC_x (pe^t)^x \cdot q^{n-x} = (q+pe^t)^n$$

2. About mean: $M_{x-np}(t) = E[e^{t(x-np)}]$

G.B.T.U. 2012
U.P.T.U. 2015

$$= e^{-npt} E(e^{tx}) = e^{-npt} M_x(t)$$

$$= e^{-npt}(q+pe^t)^n = (qe^{-pt} + pe^{t-pt})^n$$

$$= (qe^{-pt} + pe^{t-pt})^n = (qe^{-pt} + pe^{qt})^n \quad [\because 1-p=q]$$

Applications of Binomial Distribution

1. In Problems concerning no. of defectives in a sample Production line.
2. In estimation of reliability of systems.
3. No. of rounds fired from a gun hitting a target.
4. In Radar detection.

Question: A binomial variable x satisfies the relation $9P(x=4) = P(x=2)$ when $n=6$. Find the value of the Parameter p and $P(x=1)$

Solution: We know that $P(x=r) = {}^nC_r\, p^r q^{n-r}$

$P(x=4) = {}^6C_4\, p^4 q^2 = 15p^4q^2$ Since $n=6$

$P(x=2) = {}^6C_2\, p^2 q^4 = 15p^2q^4$

and The given relation is $9P(x=4) = P(x=2)$

$\Rightarrow 9 \times 15p^4q^2 = 15p^2q^4 \Rightarrow 9p^2 = q^2 = (1-p)^2$

$9p^2 = 1+p^2-2p \Rightarrow 8p^2+2p-1=0 \Rightarrow (4p-1)(2p+1)=0$

$\boxed{p = \frac{1}{4}}$, $q = 1-\frac{1}{4} = \frac{3}{4}$ $[\because p = -\frac{1}{2}$ cannot be negative$]$

Now $P(x=1) = {}^6C_1 \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^5 = .3559$

(18)

Question:→ Out of 800 families with 4 children each, How many families would be expected have (i) 2 boys and 2 girls (ii) at least one boy (iii) no girl (iv) atmost two girls? Assume equal Probabilities for boys and girls.

Solution:→ Since probabilities for boys and girls are equal,

p = Probability of having a boy $= \frac{1}{2}$, ∵ for a girl $q = \frac{1}{2}$

$n = 4$ $\quad N = 800$

(i) The expected number of families having 2 boys & 2 girls

$$= 800\left[{}^4C_2\left(\tfrac{1}{2}\right)^2\cdot\left(\tfrac{1}{2}\right)^2\right] = 800 \times 6 \times \tfrac{1}{16} = 300 \text{ Ans.}$$

(ii) The expected number of families having at least one boy

$$= 800\left({}^4C_1\left(\tfrac{1}{2}\right)^3\cdot\left(\tfrac{1}{2}\right) + {}^4C_2\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^2 + {}^4C_3\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^3 + {}^4C_4\left(\tfrac{1}{2}\right)^4\right]$$

$$= 800 \times \tfrac{1}{16}\left(4+6+4+1\right] = 750 \text{ Ans}$$

(iii) The expected number of families having no girl, $r = 4$,

$$= 800 \times {}^4C_4\left(\tfrac{1}{2}\right)^4 \times \left(\tfrac{1}{2}\right)^0 = 50 \text{ Ans}$$

(iv) The expected number of families having atmost two girls

i.e. having at leas 2 boy

$$= 800\left[{}^4C_2\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^2 + {}^4C_3\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^3 + {}^4C_4\left(\tfrac{1}{2}\right)^4\right] = 800 \times \tfrac{1}{16}\left[6+4+1\right]$$

$$= 550 \text{ Ans.}$$

Question:→ Six dice are thrown 729 times. How many times do you expect at least three dice to show a five or six?

Sol:→ p = the chance of getting 5 or 6 with on die $= \frac{2}{6} = \frac{1}{3}$

$q = 1 - \frac{1}{3} = \frac{2}{3}$, $n = 6$, $N = 729$, Since die have $n = 6$

The expected number of times at least three dice Showing five or six $= N \cdot P(r \geq 3) = 729\left(P(3) + P(4) + P(5) + P(6)\right]$

$$= 729\left[{}^6C_3\left(\tfrac{2}{3}\right)^3\left(\tfrac{1}{3}\right)^3 + {}^6C_4\left(\tfrac{2}{3}\right)^2\left(\tfrac{1}{3}\right)^4 + {}^6C_5\left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right)^5 + {}^6C_6\left(\tfrac{1}{3}\right)^6\right]$$

$$= \frac{729}{3^6}\left[160 + 60 + 12 + 1\right] = 233.$$