7.6 CAUSALITY

Definition A system is said to be a 'causal system' or a 'non-anticipatory system' if its output at any instant of time depends for its value only on the input at that instant and the previous instants but not on the input at future instants.

This only means that a causal system cannot anticipate what the future values of input would be and respond to them now. Its present response can depend on the present input and even some of the past inputs (because of its memory), but not on future inputs.

It is clear from the foregoing that all physically realizable real-time systems must be causal.

Thus, systems whose input–output relation is of the form,

$$y(t) = Ax(t+1) + B\,x(t) \text{ where } A \text{ and } B \text{ are constants}$$

or

$$y(n) = [Ax(n-1) + Bx(n) + Cx(n+1)] \text{ where } A, B \text{ and } C \text{ are constants.}$$

cannot obviously be causal since the present output $y(t)$ in t he first case, depends on $x(t+1)$—a future value of the input, and $y(n)$, the present output in the second case, depends on $x(n+1)$, the next input sample value.

Till now, we have discussed about various types of continuous-time and discrete-time systems—linear and non-linear, fixed and time-varying, static and dynamic, causal and non-causal.

However, we shall henceforth confine our discussion only to the important class of systems which are linear and also time (or shift) invariant and are called linear time-invariant (or shift-invariant) systems.

The reasons for this are the following:

(i) Analysis of non-linear systems is quite difficult.

(ii) Even if the systems that we come across in practice are not exactly linear, they can be approximated by linear systems.

(iii) The theory of linear systems is quite well developed and simple.

Hence, unless otherwise specified, hereafter, whenever we use the term 'system', either in the continuous-time context or discrete-time context, it should be understood that we are referring only to a linear time (or shift) -invariant system.

7.7 IMPULSE RESPONSE OF SYSTEMS

The impulse response, $h(t)$, of a linear time-Invariant continuous-time system, is defined as the response of the system to a unit impulse given to it as input when the system is in ground state. (see Remark 3, page 304)

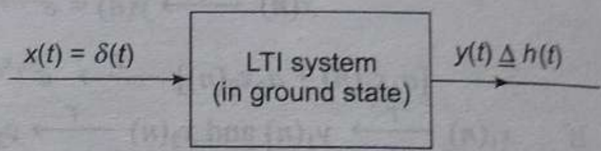


Fig. 7.12 Impulse response of a system

Unit-Sample Response Sequence The unit sample response sequence, $h(n)$, of a linear shift-invariant discrete-time system is defined as the response of the system in ground state, to a unit-sample sequence applied to it as input.

A discrete-time system is said to be in ground state if all its memory elements contain only zeros.

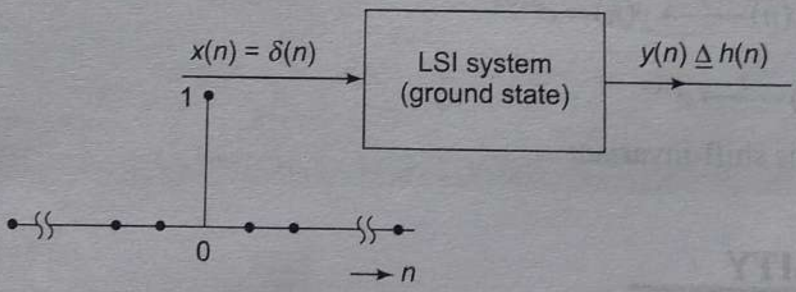


Fig. 7.13 Unit-sample response sequence

Note The term, 'impulse response', is used both for continuous-time as well as discrete-time systems. But, it is more appropriate to use the term, 'unit-sample response sequence' in the case of discrete-time systems.

The impulse-response function, $h(t)$, of an LTI system and the unit-sample response sequence, $h(n)$, of an LSI system, are analogous entities and play a key role in the description of the behavior of LTI and LSI systems respectively. We will now show that $h(t)$ (or $h(n)$) completely characterizes the system; i.e., if we know this function (or sequence), it enables us to know everything that we need to know about the electrical behavior of the system—whether it is causal or not, whether it is stable or not, what its frequency response would be, and what its output would be for any arbitrarily given input signal.

Properties of convolution and inter-connected systems

We now give some useful properties of convolution.

(i) Distributive Property $x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$

(ii) Associative Property $[h_1(t) * h_2(t)] * h_3(t) = h_1(t) * [h_2(t) * h_3(t)]$

(iii) Commutative Property $h_1(t) * h_2(t) = h_2(t) * h_1(t)$.

Proofs for the above properties are very simple and are left to the reader as an exercise.

(a) LTI systems connected in parallel

Consider two LTI systems T_1 and T_2 with impulse responses $h_1(t)$ and $h_2(t)$ connected in parallel, as shown.

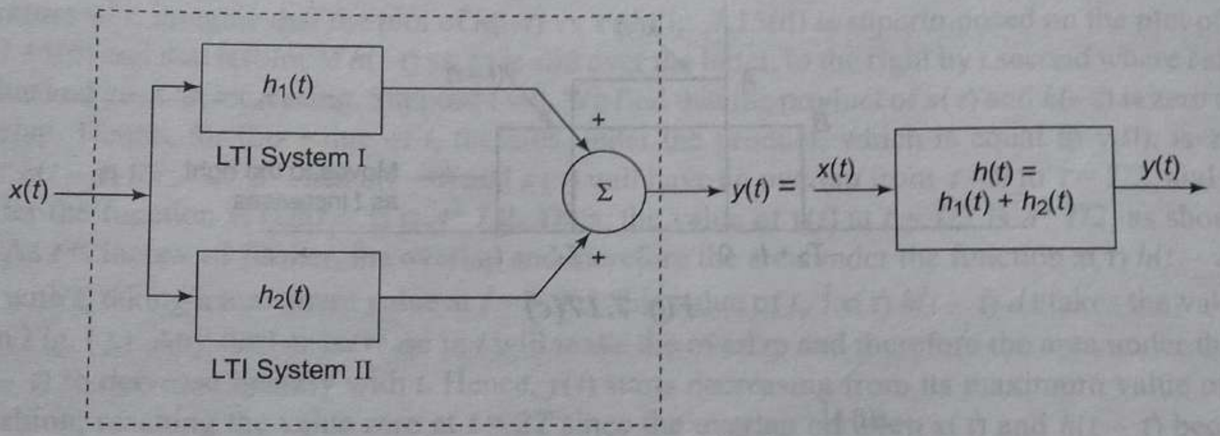


Fig. 7.18 *Systems connected in parallel*

$$y(t) = [x(t) * h_1(t)] + [x(t) * h_2(t)] = x(t) * [h_1(t) + h_2(t)] = x(t) * h(t)$$

(b) LTI systems connected in cascade

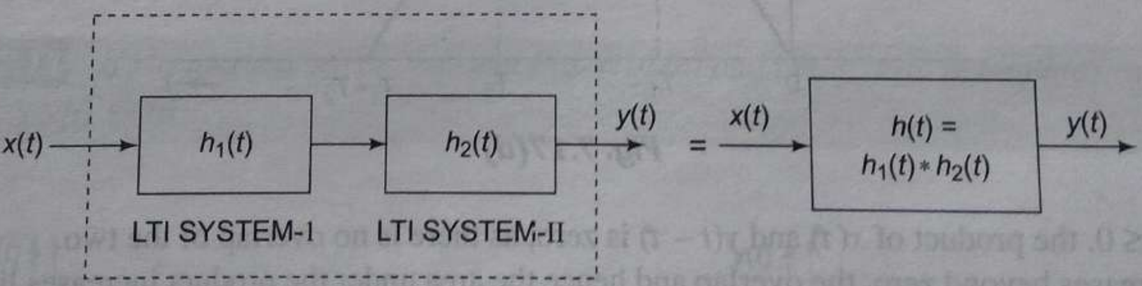


Fig. 7.19 *Systems connected in cascade*

$$y(t) = [x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

c) The commutative property tells us that the order in which we connect a number of LTI systems in cascade, is immaterial.

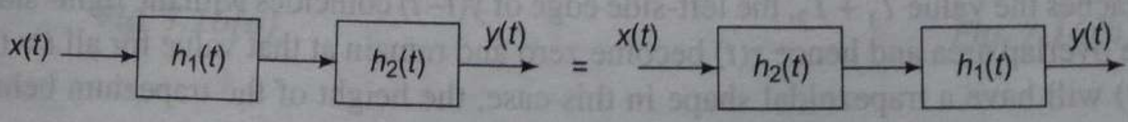


Fig. 7.20 *Interchanging the order in which systems are connected in cascade*

Example 7.12 *Show that an ideal differentiator with input x(t) and output y(t) related by* $y(t) = \frac{dx(t)}{dt}$, *is a linear time-invariant system.*

Solution We are given that $x(t) \xrightarrow{T} y(t) = \frac{dx(t)}{dt}$

Hence, if $x_1(t) \xrightarrow{T} y_1(t)$ then $y_1(t) = \frac{dx_1(t)}{dt}$

and if $x_2(t) \xrightarrow{T} y_2(t)$, then $y_2(t) = \frac{dx_2(t)}{dt}$

Also, if $[a_1x_1(t) + a_2x_2(t)]$ is given as the input,

$$[a_1x_1(t) + a_2x_2(t)] \xrightarrow{T} y(t) = \frac{d}{dt}[a_1x_1(t) + a_2x_2(t)] = a_1\frac{dx_1(t)}{dt} + a_2\frac{dx_2(t)}{dt}$$

$$\therefore \quad y(t) = a_1 y_1(t) + a_2 y_2(t)$$

Hence, the system T, i.e., the ideal differentiator, is a linear system.

To show that it is time-invariant, consider

$$x_1(t) = x(t-\tau)$$

Then $$x_1(t) \xrightarrow{T} y_1(t) = \frac{dx(t-\tau)}{dt}$$

Put $t - \tau = \lambda \therefore dt = d\lambda$ and $\frac{dx(t-\tau)}{dt} = \frac{dx(\lambda)}{d\lambda} = y(\lambda) = y(t-\tau)$

$\therefore$ The ideal differentiator is time-invariant.

Example 7.13 *Show that an ideal integrator, whose input x(t) and output y(t) are related by*

$$y(t) = \int_{-\infty}^{t} x(\lambda)d\lambda$$

is an LTI system.

Solution If $$x_1(t) \xrightarrow{T} y_1(t) = \int_{-\infty}^{t} x_1(\lambda)d\lambda \text{ and}$$

$$x_2(t) \xrightarrow{T} y_2(t) = \int_{-\infty}^{t} x_2(\lambda)d\lambda$$

Then $$[a_1x_1(t) + a_2x_2(t)] \xrightarrow{T} y(t) = \int_{-\infty}^{t} [a_1x_1(\lambda) + a_2x_2(\lambda)]d\lambda$$

$$= a_1\int_{-\infty}^{t} x_1(\lambda)d\lambda + a_2\int_{-\infty}^{t} x_2(\lambda)d\lambda$$

$$= a_1 y_1(t) + a_2 y_2(t)$$

$\therefore$ the system is linear.

To show that the ideal integrator is time-invariant,

let $$x_1(\lambda) = x(\lambda - \tau)$$

Also, let $$x_1(t) \xrightarrow{T} y_1(t) = \int_{-\infty}^{t} x_1(\lambda)d\lambda = \int_{-\infty}^{t} x(\lambda - \tau)d\lambda$$

If we put $\lambda - \tau = z$, $d\lambda = dz$ and when $\lambda = t$, $z = (t - \tau)$

$\therefore$ $$\int_{-\infty}^{t} x(\lambda - \tau)d\lambda = \int_{-\infty}^{(t-\tau)} x(z)dz = y(t - \tau)$$

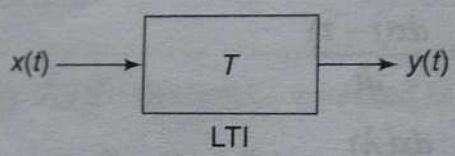
Thus, $$x(t - \tau) \xrightarrow{T} y(t - \tau)$$

Hence the system is time-invariant.

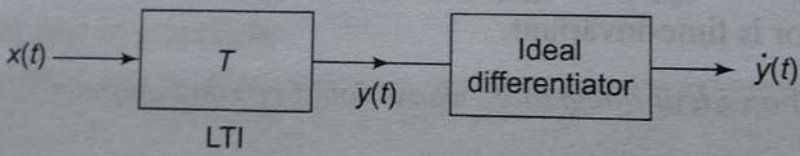
$\therefore$ an ideal integrator is an LTI system.

If we combine the results of examples 7.8 and 7.9 with the fact that two LTI systems may be connected in cascade in any order (refer to commutative property of convolution), we get the following two important results.

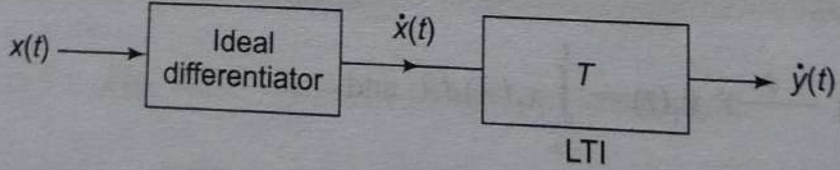
(i) Let



Consider,



Since interchanging the order of the two systems should not make any difference, the output should again be $\dot{y}(t)$.



Thus, $$\text{if } x(t) \xrightarrow{T} y(t), \text{ then } \dot{x}(t) \xrightarrow{T} \dot{y}(t) \qquad \ldots(7.11)$$

(ii) In a similar way, **if $x(t) \xrightarrow{T} y(t)$, then**

$$\int_{-\infty}^{t} x(\lambda)d\lambda \xrightarrow{T} \int_{-\infty}^{t} y(\lambda)d\lambda \qquad \ldots(7.12)$$

7.9 UNIT-SAMPLE RESPONSE CHARACTERIZATION OF LSI SYSTEMS

Just as an LTI system could be completely characterized by its impulse response function, $h(t)$, an LSI system can be shown to be completely characterized by its unit-sample response sequence, $h(n)$.

For this purpose let us first see how we may represent any arbitrary discrete-time signal, or sequence, $\{x(n)\}$, in terms of a weighted sum of an infinite number of shifted unit-sample sequences.

The zeroth sample of $x(n)$, viz., $x(0)$, may be represented as $x(0)\{\delta(n)\}$, where $\{\delta(n)\}$ is the unit-sample sequence. The next sample, i.e., the first sample, $x(1)$ which occurs at $n = 1$, can be represented by $x(1)\{\delta(n-1)\}$. Thus, the k^{th} sample, having a value of $x(k)$ and occurring at $n = k$, can be represented by $x(k)\{\delta(n-k)\}$. Hence, the sequence $\{x(n)\}$ can be written as

$$\{x(n)\} = \sum_{k=-\infty}^{\infty} x(k)\{\delta(n-k)\} \qquad \ldots (7.15)$$

If the discrete-time signal $\{x(n)\}$ is given as input to an LSI system T with $\{h(n)\}$ as its unit-sample response sequence, and if the output is $\{y(n)\}$, we can write

$$\{y(n)\} = T\{x(n)\} = T\left[\sum_{k=-\infty}^{\infty} x(k)\{\delta(n-k)\}\right]$$

$$= \sum_{k=-\infty}^{\infty} x(k)T\left[\{\delta(n-k)\}\right]$$

i.e.,

$$\{y(n)\} = \sum_{k=-\infty}^{\infty} x(k)\{h(n-k)\}$$

or,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \qquad \ldots (7.16)$$

RHS of Eq. (7.16) is called the convolution sum of the two sequences $\{x(n)\}$ and $\{h(n)\}$. A knowledge of $\{h(n)\}$ will enable one to determine the output for any input using Eq. (7.16). The USRS, $\{h(n)\}$, thus completely characterizes the LSI system.