7.2 CLASSIFICATION OF SYSTEMS

Systems may be broadly classified as continuous-time systems and discrete-time systems. These are defined as follows:

A continuous-time system is one which takes a continuous-time signal as input and produces another continuous-time signal as output.

A discrete-time system is one which takes a discrete-time signal as input and produces another discrete-time signal as output.

It must, however, be noted that every system need not necessarily come under one of the above two categories. For instance, a system may take a continuous-time signal as input, and give a discrete-time signal as output, as in the case of a sampler; or, it may take a discrete-time signal as input and give a continuous-time signal as output, as in the case of a digital-to-analog converter.

Continuous-time as well as discrete-time systems may again be classified as

i) Static (or memoryless) or dynamic (with memory)
ii) Linear or non-linear
iii) Time-varying or time-invariant

We shall now consider these in detail.

7.3 STATIC AND DYNAMIC SYSTEMS

A system is said to be static or memoryless, or instantaneous if its present output is determined *entirely by the present input only*. For example, consider a continuous-time system with input-output relationship given by

$$y(t) = Ax(t) + B; A \text{ and } B \text{ are constants.}$$

It is a static system since the output, $y(t)$, at the present instant, 't', is determined entirely by the value of the input, $x(t)$, at the present instant only.

Similarly, the discrete-time system governed by an input–output relationship of the form

$$y(n) = ax(n) + b \text{ ; } a \text{ and } b \text{ are constants}$$

is also a static system.

Or, consider the following purely resistive network.

This is a static system because the output $y(t)$ is given by

$$y(t) = \left(\frac{R_2}{R_1 + R_2}\right) x(t),$$

and is therefore determined at any instant by the value of the input $x(t)$ at that instant only. It is easy to generalize from this example and say that all purely resistive networks, however complicated they may be, are 'static systems' only.

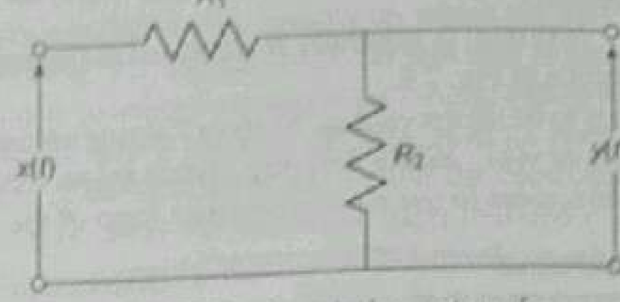

Fig. 7.8 A resistive network

But suppose we now consider the following networks.

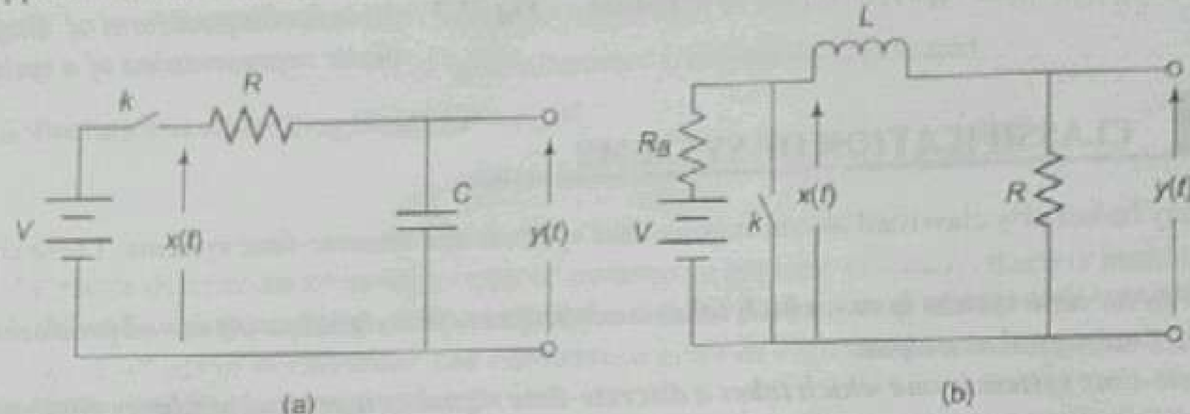

Fig. 7.9 (a) and (b) Examples of dynamic systems

For the network shown in Fig. 7.9(a), suppose we close the switch k at $t = 0$. For $t > 0$, $x(t)$ is a constant and is equal to V (neglecting internal resistance of the battery). However, for any $t > 0$, the voltage at the output, viz., $y(t)$ is dependent not only on $x(t)$ at that time, i.e., V, but also on t and the initial voltage across the capacitor.

Similarly, for the network in Fig 7.9 (b), if we allow sufficient time for the current in the inductor to reach a steady value and then close the switch k, say at time $t = 0$, then the output voltage $y(t)$ for any finite $t > 0$ is not going to be zero even though $x(t) = 0$ for $t > 0$. This is because the current through the inductance cannot change suddenly and become zero. It takes some time.

The networks of Figs. 7.9 (a) and (b) therefore represent dynamic systems, systems with memory. What makes these networks dynamic and the one in Fig. 7.8 static? It is the presence of the energy storage devices—capacitor and inductance, which has made the difference. A capacitor cannot change its voltage suddenly and an inductor cannot change its current suddenly. This is because, a sudden change in the voltage across a capacitor or a sudden change of current through an inductor, implies a sudden change in the energy stored. Instantaneous change in stored energy implies an infinite flow of power, which is impossible.

Remark 1 It is easy to show, by writing down Kirchhoff's loop equations, that the systems shown in Figs. 7.9 (a) and (b) have their input and output related through the following differential equations:

$$x(t) = RC\frac{dy(t)}{dt} + y(t) \quad \text{for system in Fig. 7.9 (a)}$$

$$x(t) = \frac{L}{R}\frac{d}{dt}y(t) + y(t) \text{ for the system in Fig. 7.9(b).}$$

and

Discrete-Time Systems

A discrete-time system described by an input–output relationship

$$y(n) = A\,x(n) + B \text{ ; } A \text{ and } B \text{ are constants}$$

is also a static, or memoryless system, as its present output $y(n)$ depends for its value *only on the present input*, viz., $x(n)$. This system may be realized as shown in Fig. 7.10.

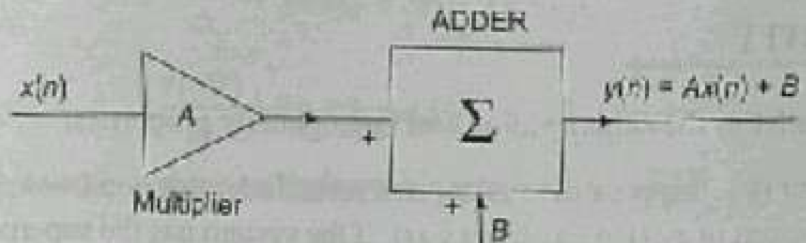

Fig. 7.10 A static discrete-time system

On the other hand, a discrete-time system with input–output relationship given by

$$y(n) = ax(n) - bx(n-1) \text{ , } a \text{ and } b \text{ are constants} \qquad \ldots (7.3)$$

is a dynamic system, or a system with memory, because the present output $y(n)$ depends not just on the present input, $x(n)$, but also on a past input $x(n-1)$. This system may be realized as shown in Fig. 7.11.

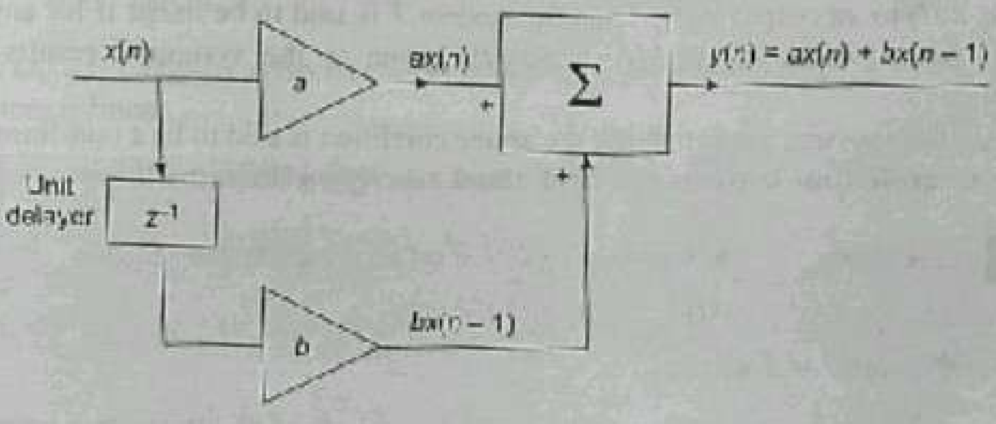

Fig. 7.11 Realization of the system described by Eq. 7.3

Remark 2 In general, a discrete-time system, whose input–output relationship is in the form of a difference equation, is a dynamic system, as the present output depends on one or more previous inputs, and its realization involves memory units (unit delayers).

From remarks 1 and 2, we find that the input–output relationship for dynamical systems which can be realized through physical circuits can be expressed in the form of a differential equation in the case of continuous-time systems and in the form of a difference equation in the case of discrete-time systems.

However, for static or instantaneous systems, the input–output relation must be of the form

$$y(t) = f[x(t), t] \text{ for continuous-time system}$$

and $$y(n) = f[x(n), n] \text{ for discrete-time systems}$$

where $f[.]$ is a function which depends on $x(t)$ (or $x(n)$) only at the present instant t (or n).

Remark 3

a) A continuous-time dynamic system is said to be 'at rest', or 'in ground state', if all of its energy storage elements are devoid of any stored energy.

b) A discrete-time dynamic system is said to be 'at rest', or 'in ground state', if the contents of all its memory elements are zeros.

7.4 LINEARITY

A system is linear if it satisfies the superposition and homogeneity properties.

Superposition property Suppose the system is at rest. Under this condition, let an input $x_1(t)$ give rise to an output $y_1(t)$ and an input of $x_2(t)$ to an output $y_2(t)$. If the system has the superposition property, then, an input of $[x_1(t) + x_2(t)]$ should give rise to an output of $[y_1(t) + y_2(t)]$.

Homogeneity Property Suppose the system is at rest and that under that condition an input signal $x(t)$ gives rise to an output signal $y(t)$. If the system has the homogeneity property, an input signal of $a \cdot x(t)$ should give rise to an output signal of $a \cdot y(t)$, where 'a' is any arbitrary constant.

A definition of linearity which takes care of both these properties is as follows.

Definition Let T be a continuous-time system which is at rest. Let an input $x_1(t)$ to T give rise to an output $y_1(t)$ and an input $x_2(t)$ to an output $y_2(t)$. Then the system T is said to be linear if for any pair of arbitrary constants a_1 and a_2, an input of $[a_1x_1(t) + a_2x_2(t)]$ given to the system T results in an output of $[a_1y_1(t) + a_2y_2(t)]$.

Any continuous-time system not satisfying the above condition is said to be a non-linear system.

Linearity for discrete-time systems can be defined exactly on the same lines.

Example 7.1 *A particular system has been modeled by an input–output relation*

$$y(t) = a_0 + a_1x(t) + a_2x^2(t)$$

(i) *Is this system static or dynamic?*

(ii) *Is it linear?*

Justify your answers.

Solution

(i) From the input–output relation, we find that the output $y(t)$ at the instant 't' depends for its value only on the input $x(t)$ at that instant. Hence the system is instantaneous, memoryless or static.

(ii)
$$x_1(t) \xrightarrow{T} y_1(t) = a_0 + a_1x_1(t) + a_2x_1^2(t)$$

$$x_2(t) \xrightarrow{T} y_2(t) = a_0 + a_1x_2(t) + a_2x_2^2(t)$$

$$\therefore [\alpha_1x_1(t) + \alpha_2x_2(t)] \xrightarrow{T} a_0 + a_1[\alpha_1x_1(t) + \alpha_2x_2(t)] + a_2[\alpha_1x_1(t) + \alpha_2x_2(t)]^2 \neq \alpha_1y_1(t) + \alpha_2y_2(t)$$

Hence, the given system is not a linear system.

Example 7.2 *A continuous-time system is described by the following differential equation.*

$$2\frac{dy(t)}{dt}+5y(t)=x(t)$$

Is this system linear? Justify your answer.

Solution Let $y_1(t)$ be the output for an input of $x_1(t)$

and $y_2(t)$ be the output for an input of $x_2(t)$.

Hence,

$$2\frac{dy_1(t)}{dt}+5y_1(t)=x_1(t)$$

$$2\frac{dy_2(t)}{dt}+5y_2(t)=x_2(t)$$

Multiplying the above two equations (on both sides) by α_1 and α_2 respectively,

$$2\alpha_1\frac{dy_1(t)}{dt}+5\alpha_1y_1(t)=\alpha_1x_1(t)$$

$$2\alpha_2\frac{dy_2(t)}{dt}+5\alpha_2y_2(t)=\alpha_2x_2(t)$$

Adding the two equations,

$$2\left[\frac{d}{dt}\{\alpha_1y_1(t)+\alpha_2y_2(t)\}\right]+5[\alpha_1y_1(t)+\alpha_2y_2(t)]=\alpha_1x_1(t)+\alpha_2x_2(t)$$

$\therefore$ An input of $\alpha_1x_1(t)+\alpha_2x_2(t)$ gives rise to an output of $\alpha_1y_1(t)+\alpha_2y_2(t)$

Hence the system is linear.

Example 7.3 *A simple discrete-time differentiator is given by*

$$y(n)=[\,x(n)-x(n-1)\,]$$

Is this system

(i) *static or dynamic?* (ii) *linear or non-linear?*

Solution

(i) As the present output $y(n)$ depends for its value not only on the present input $x(n)$, but also on one previous input, viz., $x(n-1)$, the system is dynamic.

(ii) Let $x_1(n)\xrightarrow{T}y_1(n)$ and $x_2(n)\xrightarrow{T}y_2(n)$

$$a_1y_1(n)=a_1\{x_1(n)-x_1(n-1)\}$$

and

$$a_2y_2(n)=a_2\{x_2(n)-x_2(n-1)\}$$

$\therefore$ If $[a_1x_1(n)+a_2x_2(n)]$ is given as input to the system, the corresponding output is given by

$$y(n)=\{[a_1x_1(n)+a_2x_2(n)]-[a_1x_1(n-1)+a_2x_2(n-1)]\}$$

$$=a_1[x_1(n)-x_1(n-1)]+a_2[x_2(n)-x_2(n-1)]$$

$$= a_1y_1(n) + a_2y_2(n)$$

Hence, the system is linear.

7.5 TIME-INVARIANCE

Time-invariance is the property of a system which makes the behavior of the system independent of time. This means that the behavior of the system does not depend on the time at which the input is applied. In other words, it means that the output or response of the system will be delayed exactly by the same amount of time as the amount of time by which the input signal itself is delayed. Systems possessing this property are called 'time-invariant' or 'fixed' systems.

A precise definition of a time-invariant continuous-time system is as follows.

Definition Let $y(t)$ be the response of a continuous-time system T to an arbitrary input signal $x(t)$. Then the system T is said to be 'time-invariant' or 'fixed', if, for any value of the real constant τ, it gives a response of $y(t-\tau)$ for an input of $x(t-\tau)$. If this condition is not satisfied, the system T is said to be a 'time-varying' system.

Since a discrete-time signal can be delayed or advanced only by an integer number of sampling intervals, we generally call the property 'shift-invariance' in the case of discrete-time systems and define it as follows.

Definition Let $y(n)$ be the output signal of a discrete-time system T to an arbitrary discrete-time signal $x(n)$. T is said to be a 'fixed discrete-time system' or simply, a 'shift-invariant system', if for any integer n_0, the response of the system for an input of $x(n-n_0)$ is equal to $y(n-n_0)$.

Remark It must be noted that since the properties of 'instantaneity', 'linearity ' and 'time-invariance' are independent of each other, a system possessing one property need not necessarily possess the other properties.

Hence, a linear system may be static or dynamic and can be fixed or time-varying. Similarly, a time-invariant system may be static or dynamic and may be linear or non-linear.

Example 7.4 *The input–output relation for a continuous-time system, is given by*

$$y(t) = x(2t)$$

Is this system

(i) *static or dynamic?* (ii) *linear or nonlinear?* (iii) *fixed or time-varying?*

Justify your answers.

Solution

(i) Since $y(t) = x(2t)$, the output, at any instant of time t_1 depends for its value on the present input for $t_1 = 0$, on future values of input for $t_1 > 0$ and on past values of input for $t_1 < 0$. Hence, the system is not static.

(ii)
$$x(t) \xrightarrow{T} x(2t) = y(t)$$

$\therefore$
$$x_1(t) \xrightarrow{T} x_1(2t) = y_1(t)$$

and $$x_2(t) \xrightarrow{T} x_2(2t) = y_2(t)$$

$$\therefore [a_1x_1(t)+a_2x_2(t)] \xrightarrow{T} [a_1x_1(2t)+a_2x_2(2t)] = a_1y_1(t)+a_2y_2(t)$$

$\therefore T$ is linear.

(iii)
$$x(t) \xrightarrow{T} x(2t) = y(t)$$

$$x(t-\tau) \xrightarrow{T} x(2t-\tau) \neq y(t-\tau) \text{ since } y(t-\tau) = x(2t-2\tau)$$

$\therefore T$ is time-variant, i.e., it is not fixed.

Example 7.5 *The input and output of a discrete-time system are related as follows.*

$$y(n) = x(2n)$$

Is this system

(i) *static or dynamic?* (ii) *linear?* (iii) *fixed or time-varying?*

Solution

(i) The present value of the output at n, viz., $y(n)$ depends for its value on future / past values of the input; on future values when $n > 0$ and past values when $n < 0$. Hence the system is dynamic.

(ii)
$$x_1(n) \xrightarrow{T} x_1(2n) = y_1(n)$$

$$x_2(n) \xrightarrow{T} x_2(2n) = y_2(n)$$

and $$[a_1x_1(n)+a_2x_2(n)] \xrightarrow{T} [a_1x_1(2n)+a_2x_2(2n)] = a_1y_1(n)+a_2y_2(n)$$

$\therefore$ the system is linear.

(iii)
$$x(n) \xrightarrow{T} x(2n) = y(n)$$

$$x(n-n_0) \xrightarrow{T} x(2n-n_0) \neq y(n-n_0) \text{ since } y(n-n_0) = x(2\overline{n-n_0}) = x(2n-2n_0)$$

$\therefore$ the system is time-variant, or is not fixed.

Example 7.6 *Is the simple discrete-time differentiator of Example 7.3 fixed or shift-variant?*

Solution

$$y(n) = [x(n) - x(n-1)]$$

$$x(n) \xrightarrow{T} [x(n)-x(n-1)] = y(n)$$

$$x(n-n_0) \xrightarrow{T} [x(n-n_0)-x(n-n_0-1)] = y(n-n_0)$$

$\therefore$ the system is fixed.

Example 7.7 *A discrete-time system is described by the following input–output relation:*

$$y(n) = n\,x(n)$$

Is this system

(i) *static or dynamic?* (ii) *linear or non-linear?* (iii) *time-varying or time-invariant?*

Give justifications for your answers.

Solution

(i) The system is static since the present output at the instant 'n' depends only on the present input $x(n)$.

(ii) If $x_1(n)$ is given as input,

$$x_1(n) \xrightarrow{T} nx_1(n) \triangleq y_1(n)$$

If $x_2(n)$ is given as input,

$$x_2(n) \xrightarrow{T} nx_2(n) \triangleq y_2(n)$$

$$\therefore \quad [a_1x_1(n) + a_2x_2(n)] \xrightarrow{T} n[a_1x_1(n) + a_2x_2(n)]$$
$$= a_1[nx_1(n)] + a_2[nx_2(n)]$$
$$= a_1y_1(n) + a_2y_2(n)$$

Hence, the system is linear.

(iii) We have $x(n) \xrightarrow{T} nx(n) \triangleq y(n)$

If $x(n-n_0)$, where n_0 is an integer, is given as input,

$$x(n-n_0) \xrightarrow{T} nx(n-n_0)$$

But, since $y(n) \triangleq nx(n)$,

$$y(n-n_0) = (n-n_0)x(n-n_0)$$

$\therefore$ The response of T for $x(n-n_0)$ as input, is not equal to $y(n-n_0)$. Therefore the system is time-varying.

Example 7.8 *A discrete-time system is described by the following input–output relationship*

$$y(n) = \sum_{k=n-2}^{n+2} x(k)$$

Check whether the system is (i) linear, and (ii) shift-invariant.

Solution

(i)
$$x_1(n) \xrightarrow{T} y_1(n) = \sum_{k=n-2}^{n+2} x_1(k)$$

$$x_2(n) \xrightarrow{T} y_2(n) = \sum_{k=n-2}^{n+2} x_2(k)$$

$$\therefore [a_1x_1(n) + a_2x_2(n)] \xrightarrow{T} y(n) = \sum_{k=n-2}^{n+2} [a_1x_1(k) + a_2x_2(k)]$$

$$= a_1 \sum_{k=n-2}^{n+2} x_1(k) + a_2 \sum_{k=n-2}^{n+2} x_2(k)$$
$$= a_1y_1(n) + a_2y_2(n)$$

$\therefore$ The given system is linear.

(ii) $$x(n-m) \xrightarrow{T} \sum_{k=n-2}^{n+2} x(k-m)$$

But $$y(n-m) = \sum_{k=n-m-2}^{n-m+2} x(k) = \sum_{k=n-2}^{n+2} x(k-m)$$

$\therefore$ $$x(n-m) \xrightarrow{T} y(n-m)$$

$\therefore$ the system is shift-invariant.

Example 7.9 *Check whether the following system is (i) linear, and (ii) shift-invariant.*

$$y(n) = e^{|x(n)|}$$

Solution

$$x(n) \xrightarrow{T} y(n) = e^{|x(n)|}$$

$\therefore$ $$[a_1x_1(n) + a_2x_2(n)] \xrightarrow{T} e^{|a_1x_1(n)+a_2x_2(n)|}$$

If $x_1(n) \xrightarrow{T} y_1(n)$ and $x_2(n) \xrightarrow{T} y_2(n)$,

$$a_1y_1(n) + a_2y_2(n) = a_1e^{|x_1(n)|} + a_2e^{|x_2(n)|}$$

$$\neq e^{|a_1x_1(n)+a_2x_2(n)|}$$

Hence, the system is not linear.

(ii) Shift-invariance $x(n) \xrightarrow{T} y(n) = e^{|x(n)|}$

$\therefore$ $$x(n-m) \xrightarrow{T} e^{|x(n-m)|} = y(n-m)$$

Hence, the system is shift-invariant