Example 6.6. Find the Nyquist rate and Nyquist interval for the continuous-time signal given below :

$$x(t) = \frac{1}{2\pi} \cos(4000\pi t) \cos(1000\pi t)$$

Solution : Given that

$$x(t) = \frac{1}{2\pi} \cos(4000\pi t) \cos(1000\pi t)$$

The above equation may be expressed as

or $$x(t) = \frac{1}{4\pi} [2 \cos(4000\pi t) \cos(1000\pi t)$$

or $$x(t) = \frac{1}{4\pi} [\cos(4000\pi t + 1000\pi t) + \cos(4000\pi t - 1000\pi t)]$$

or $$x(t) = \frac{1}{4\pi} [\cos(5000\pi t) + \cos(3000\pi t)] \quad ...(i)$$

Now let us assume that there are two frequencies present in the signal. This signal may be represented as

$$x(t) = A_1 \cos \omega_1 t + A_2 \cos \omega_2 t \quad ...(ii)$$

where A_1 and A_2 are the amplitudes, ω_1 and ω_2 are the angular frequencies. Comparing equation (*i*) with equation (*ii*), we get

$$\omega_1 = 5000\pi \text{ radian/sec}$$
$$\omega_2 = 3000\pi \text{ radian/sec}$$

Therefore, $$f_1 = \frac{\omega_1}{2\pi}$$

or $$f_2 = \frac{5000\pi}{2\pi} = 2500 \text{ Hz}$$

$$f_2 = \frac{\omega_2}{2\pi}$$

or $$f_2 = \frac{3000\pi}{2\pi} = 1500 \text{ Hz}$$

Then, highest frequency component of the given meassage signal will be

$$f_{max} = \text{Max } \{f_1, f_2\}$$
$$= \text{Max } \{2500, 1500\} = 2500 \text{ Hz}$$

Hence, Nyquist rate $= 2 \times f_{max}$
$= 2 \times 2500 = 5000$ Hz

and, Nyquist interval $= \dfrac{1}{\text{Nyquist rate}}$

$$= \frac{1}{5000} = 2 \times 10^{-4} = 200 \text{ milliseconds} \quad \textbf{Ans.}$$

Example 6.7. A real-valued continuous-time signal $x(t)$ is known to be uniquely determined by its samples when the sampling frequency is $\omega_s = 10^4\ \pi$ radian/sec. Find the value of ω when $X(j\omega)$ is guaranteed to be zero.

Solution : From sampling theorem, we know that for perfect reconstruction of the message signal from its samples, sampling frequency or sampling rate should be greater than twice the highest frequency component of the message signal.

If $$X(j\omega) = 0 \text{ for } \omega| > \frac{\omega_s}{2}$$

then the signal is recovered perfectly from its samples.

Hence, $$\text{CTFT } \{x(t)\} = X(j\omega)$$

$$= 0 \text{ for } w| > \frac{w_s}{2}$$

or $$\omega| > \frac{10^4\pi}{2}$$

or $$|\omega| > 5 \times 10^3\pi \quad \textbf{Ans.}$$

Example 6.8. A continuous-time signal $x(t)$ is obtained at the output of an ideal low pass filter with cutoff frequency $\omega_c = 10^3\ \pi$ radian/sec. If impulse-train sampling is performed on signal $x(t)$, which of the following sampling periods will guarantee that $x(t)$ may be recovered from its sampled version using an appropriate low pass filter?

(*i*) $T_s = 5 \times 10^{-4}$ **seconds**

(*ii*) $T_s = 2 \times 10^{-3}$ **seconds**

(*iii*) $T_s = 1 \times 10^{-4}$ **seconds**

Solution : From sampling theorem, we know that the sampling rate must be atleast equal to twice the highest frequency component, *i.e.*, $2\omega_c$.

Thus, Sampling frequency $\geq 2\omega_c$

$$\geq 2 \times 10^3 \pi \text{ radian/secs} \quad ...(i)$$

It is given that

(i) $$f_s = \frac{1}{T_s}$$

or $$f_s = \frac{1}{5 \times 10^{-4}} = 2 \times 10^3 \text{ Hz}$$

or $$\omega_s = 2\pi f_s$$

$$\omega_s = 4\pi \times 10^3 \text{ radian/sec}$$

or $$\omega_s = 4 \times 10^3 \pi \text{ radian/sec} \quad ...(ii)$$

(ii) Again, $$f_s = \frac{1}{T_s} = \frac{1}{2 \times 10^{-3}} = 500 \text{ Hz}$$

$$\omega_s = 2\pi f_s$$

$$\omega_s = 2\pi \times 500 = 10^3 \pi \text{ radian/sec} \quad ...(iii)$$

(iii), Again $$f_s = \frac{1}{T_s} = \frac{1}{1 \times 10^{-4}} = 10^4 \text{ Hz}$$

$$\omega_s = 2\pi f_s = 2\pi \times 10^4 = 2 \times 10^4 \pi \text{ radian/secs} \quad ...(iv)$$

From Eqations (i), (ii), (iii) and (iv), we clearly know that only conditions (i) and (iii) satisfy the sampling theorem condition. **Ans.**

Example 6.9. Find the Nyquist rate for the continuous-time signal given below

$$x(t) = \frac{\sin(4 \times 10^3 \pi t)}{\pi t}$$

Solution: We know that

$$\text{CTFT}\left[\frac{\sin(\tau t)}{\pi t}\right] = X(j\omega)$$

$$= \begin{cases} 1, |\omega| < \tau \\ 0, |\omega| > \tau \end{cases} \quad ...(i)$$

Substituting $\tau = 4 \times 10^3 \pi$ radian/sec in equation (i), we get

$$CTFT\left[\frac{\sin(4 \times 10^3 \pi t)}{\pi t}\right] = X(j\omega)$$

$$= \begin{cases} 1, |\omega| < 4 \times 10^3 \pi \\ 0, |\omega| > 4 \times 10^3 \pi \end{cases} \quad ...(ii)$$

It implies that

$$\text{CTFT } \{x(t)\} = X(j\omega)$$

$$= 0 \text{ for } |\omega| > 4 \times 10^3 \pi$$

Hence, the Nyquist rate for this signal will be

$$\omega_n = 2(4 \times 10^3 \pi)$$

$$= 8 \times 10^3 \pi \text{ radian/sec} \quad \textbf{Ans.}$$

Example 6.10. Given a continuous-time signal $x(t)$ with Nyquist rate ω_0. Determine the Nyquist rate for the continuous-time signal $x^2(t)$.

Solution : If the continuous-time signal $x(t)$ has a Nyquist rate of ω_0, then its CTFT,

$$X(j\omega) = 0 \text{ for } |\omega| > \frac{\omega_0}{2}.$$

Let us assume that $y(t) = x^2(t)$.

We also know that

$$y(t) = x^2(t) \xleftrightarrow{CTFT} Y(j\omega)$$

$$= \frac{1}{2\pi}[X(j\omega) \otimes X(j\omega)]$$

It is clear that we can only guarantee that $Y(j\omega) = 0$ for $|\omega| > \omega_0$. Hence, the Nyquist rate for $y(t)$ is $2\omega_0$. **Ans.**

Example 6.11. Given a continuous-time signal $x(t)$ with Nyquist rate ω_0. Determine the Nyquist rate for the continuous-time signal.

$$y(t) = x(t) \cos \omega_0 t$$

Solution: If the continuous-time signal $x(t)$ has the Nyquist rate of ω_0, then its

CTFT $X(j\omega) = 0$ for $|\omega| > \frac{\omega_0}{2}$.

We also know that

$$y(t) = x(t)\cos\omega_0 t \xleftrightarrow{CTFT} = \frac{1}{2}X[j(\omega - \omega_0)] + \frac{1}{2}X[j(\omega + \omega_0)]$$

It is clear that we can guaragntee that

$$Y(j\omega) = 0 \text{ for } |\omega| > \omega_0 + \frac{\omega_0}{2}$$

or
$$Y(j\omega) > \frac{3\omega_0}{2} \qquad ...(i)$$

Thus, the Nyquist rate for signal $y(t)$ is $2 \times \frac{3\omega_0}{2}$ *i.e.*, $3\omega_0$. **Ans.**

Example 6.12. (*i*) What is aliasing ? What can be done to reduce aliasing ?

(*ii*) Determine the Nyquist sampling rate and Nyquist sampling interval for the following.

(1) S_a (100 πt)

(2) $S_a(100\ \pi t) + S_a$ (50 πt) ***(U.P. Tech Sem Examination 2001-2002)***

Solution : Aliasing :

Aliasing is an effect of under sampling. Alasing occurs when sampling rate $\omega_s < 2\ \omega_m$ is not sufficiently high to prevent the shifting of high frequency information into lower frequencies. Such type of pentration of information form one band of frequencies to another is called aliasing and the resulting frequency response is called an aliased representation of the original continuous-time signal. Thus, aliasing phenomenon is defined as phenomenon of high frequency components in a spectrum of a continuous time signal seemingly taking on the identify of a lower frequency in the spectrum of its sampled version.

There are two corrective measures which are used to elimiate alising.

(*i*) A pre-alias low pass filter is used before sampling for attenuating those high frequencies that are not essential for the transmission of information.

(*ii*) A pre-alias low pass filtered signal is sampled at a rate of frequency slightly higher then the Nyquist rate.

(*ii*) (1) Given $\qquad x(t) = S_a(100\ \pi t)$

Taking Fourier transform, we get

$$F\{x(t)\} = \frac{1}{100} \text{rect} \frac{\omega}{200\pi}$$

so the max frequency component in signal will be

$$\omega_m = 100\pi \text{ rad/s}$$

so the Nequist rate $= \omega_s - 2\omega_m$

$$= 200\pi \text{ rad/s}$$

Also, sampling interval,

$$T_s = \frac{2\pi}{\omega_s}$$

or

$$T_s = \frac{2\pi}{2\omega_m} = \frac{1}{100}$$

$$T_s = 0.01 \text{ sec} \qquad \textbf{Ans.}$$

2.

$$x(t) = S_a(100\,\pi t) + S_a\,(50\,\pi^2 t)$$

Taking Fourier transform, we get

$$F[x(t)] = \frac{1}{100} rect \frac{\omega}{200\pi} + \frac{1}{50} rect \frac{\omega}{50\pi}$$

so the max frequency component in signal will be

$$\omega_m = 100\pi$$

Nequist Rate

$$= \omega_s = 2\omega_m = 200\pi$$

Sampling interval $T_s = \frac{2\pi}{\omega_s}$

$$= \frac{1}{100} \text{ sec} = 0.01 \text{ sec} \qquad \textbf{Ans.}$$

Example 6.13. Two signals $x_1(t) = \cos 20\pi t$ and $x_2(t) = \cos 100\pi t$ are sampled with sampling frequency 40 Hz. Obtain the associated time signals $x_1(n)$ and $x_2(n)$ and comment on the result.

Solution : (*i*) Given signal is

$$x_1(t) = \cos 20\pi t \qquad \ldots(i)$$

Comparing equation (*i*) with standard equation

$$x_1(t) = \cos 2\pi f_1 t, \text{ we have} \qquad \ldots(ii)$$

$$\therefore \quad 2\pi f_1 = 20\,\pi \Rightarrow f_1 = 10 \text{ Hz.}$$

Now discrete signal $x_1(n)$ is obtained by replacing 't' in equation (*ii*) by $\frac{n}{f_s}$

here f_s = Sampling frequency = 40 Hz.

Thus, equation (*ii*) becomes,

$$x_1(n) = \cos 2\pi f_1 \cdot \frac{n}{f_s}$$

or

$$x_1(n) = \cos 2\pi \frac{10n}{10}$$

or

$$x_1(n) = \cos 2\pi\left(\frac{1}{4}\right) . n = \cos\frac{\pi}{2} n$$

(ii) The given signal is,

$$x_2(t) = \cos 100\,\pi t \quad \ldots(iii)$$

Comparing it with

$$x_2(t) = \cos 2\pi f_2 t, \text{ we have}$$

$$2\pi f_2 = 100\pi \Rightarrow f_2 = 50 \text{ Hz} \quad \ldots(iv)$$

Now, discrete time signal $x_2(n)$ is obtained by substituting, $t = \frac{n}{f_s}$.

Thus, equation (iv) becomes,

$$x_2(n) = \cos 2\pi \frac{50n}{40}$$

or

$$x_2(n) = \cos 2\pi \left(\frac{5}{4}\right) n$$

or

$$x_2(n) = \cos 2\pi\left(1 + \frac{1}{4}\right) n = \cos\left(2\pi n + 2\pi \cdot \frac{1}{4} n\right) \quad \ldots(v)$$

Now, we have $\cos(2\pi n + \theta) = \cos\theta$. Therefore, equation (v) becomes,

$$x_2(n) = \cos 2\pi \cdot \frac{1}{2} n = \cos \frac{\pi}{2} n$$

Comment: *Given sampling frequency, $f_s = 40$ Hz. Hence, the frequency contained in signal should be less than or equal to $\frac{f_s}{2}$, that means ≤ 20 Hz. But is not the case in this example. Therefore, aliasing takes place. Here, both the sequences $x_1(n)$ and $x_2(n)$ are equal; due to aliasing effect.*

Example 6.14. An analog signal is given as follows:

$x_a(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$.

Find Nyquist rate.

Solution : The given equation can be written as under:

$$x_a(t) = 3\cos(2\pi \times 25t) + 10\cos(2\pi \times 150t) - \cos(2\pi \times 50t) \quad \ldots(i)$$

Now we can write,

$$x_a(t) = 3\cos(2\pi f_1 t) + 10\cos(2\pi f_2 t) - \cos(2\pi f_3 t) \quad \ldots(ii)$$

Comparing equations (i) and (ii) we get,

$$f_1 = 25 \text{ Hz}, f_2 = 150 \text{ Hz and } f_3 = 50 \text{ Hz}.$$

Thus $\quad f_{max} = 150$ Hz.

Now $\quad$ Nquist rate $= 2f_{max} = 2 \times 150$ Hz

or $\quad$ Nyquist rate $= 300$ Hz. **Ans.**

Example 6.15. A continuous time sinusoidal $x_a(t)$ with fundamental period $T_m = \frac{1}{f_m}$ is sampled at the rate $\frac{1}{T}$ to produce discrete-time sinusoid $x(n) = x_a(nt)$.

(i) Show that $x(n)$ is periodic if $\frac{T}{T_m} = \frac{k}{N}$, where k and N are integers.

Solution : (i) For the discrete time signal, the normalized frequency f_0 is given by,

$$f_0 = \frac{\text{fundamental frequency } (f_m)}{\text{sampling frequency } (f_s)} \quad \ldots(i)$$

and $$f_m = \frac{1}{T_m}$$

$$f_s = \frac{1}{T} \text{ where T is sampling time.}$$

Thus, equation (*i*) becomes

$$f_0 = \frac{1/T_m}{1/T} = \frac{T}{T_m} \quad ...(ii)$$

Now, the discrete time sequence can be expressed as under:

$$x(n) = A\cos(2\pi f_0 n + \theta) \quad ...(iii)$$

Here f_0 = Normalized frequency

and θ = Phase shift.

We know that a discrete time sequence is periodic if,

$$x(n) = x(n+N) \quad \text{for all } n \quad ...(iv)$$

Now, replacing 'n' by $n + N$ in equation (*iii*), we have

$$x(n+N) = A\cos[2\pi f_0(n+N) + \theta]$$

or $$x(n+N) = A\cos(2\pi f_0 N + 2\pi f_0 n + \theta] \quad ...(v)$$

For periodicity, we want $x(n) = x(n+N)$.

Thus, equating equations (*iii*) and (*v*) we get,

$$A\cos(2\pi f_0 n + \theta) = A\cos(2\pi f_0 N + 2\pi f_0 n + \theta) \quad ....(vi)$$

Equation (*vi*) is satisfied if $2\pi f_0 N$ is interger multiple of 2π.

That means $2\pi f_0 N = 2\pi k$; where 'k' is an integer

$$\therefore \quad f_0 = \frac{2\pi k}{2\pi N}$$

or $$f_0 = \frac{k}{N} \quad ...(vii)$$

Hence from equations (*ii*) and (*vii*), we have

$$f_0 = \frac{k}{N} = \frac{T}{T_m}$$

(*ii*) Now, the fundamental period will be

$$\frac{1}{f_0} = \frac{N}{k} \quad \textbf{Ans.}$$

Example 6.16. Determine bit rate and resolution in sampling of a signal with dynamic range of 1 Volt if the sampling rate is 20 samples/sec and an 8 bit ADC is used. What is the maximum frequency that can be present in the discrete signal.

Solution : (*i*) Here, 8 bit ADC is used.

Now bit rate = samples/sec × bits/sample

The sampling frequency = 20 samples/sec.

Therefore, bit rate = 20 × 8

or bit rate = 160 bits/sec.

(*ii*) Now, resolution is given by,

$$D = \frac{x_{max} - x_{min}}{L-1} \quad ...(i)$$

where L = number of levels in the quantization process which is related to number of bits per levels as,

$$b \geq \log_2 L$$

Here, $b = 8$ bits per level since 8 bit ADC is used. ...(*ii*)

Considering equality sign we get,

$$b = \log_2 L$$

Therefore, $L = 2^b = 2^8 = 256$ ...(*iii*)

Now, $x_{max-min}$ = dynamic range = 1 V given

Substituting these values in Equation (*i*) we get,

$$\Delta = \frac{1}{256-1} = 3.92 \times 10^{-3} \text{ Ans.}$$

(*iii*) The maximum range of frequencies that can be represented by a discrete-time signal is from $-\frac{f_s}{2}$ to $+\frac{f_s}{2}$.

Thus highest frequency is given by

$$f_{max} = \frac{f_s}{2} = \frac{20}{2} = 10 \text{ samples/sec. Ans.}$$

Example 6.17. A digital communication link carries binary coded words representing samples of input signal

$$x(t) = 3 \cos 600\pi t + 2 \cos 1800 \pi t$$

The link is operated at 10,000 bits/s and each input sample is quantized into 1024 different voltage levels.

(*i*) What is the sampling frequency and foliding frequency in Hz ?

(*ii*) What is Nyquist rate of sampling for $x(t)$ in Hz ?

(*iii*) What is resolution of quanitzation ?

Solution : (*i*) Given that

$$x(t) = 3 \cos 600 \pi t + 2 \cos 1800 \pi t \quad ...(i)$$

$\therefore$ $2\pi f_1 = 600\pi$ and $2\pi f_2 = 1800\pi$

$\therefore$ $f_1 = 300$ Hz and $f_2 = 900$ Hz.

Therefore, Maximum frequency $= f_{max} = 900$ Hz

$\therefore$ Sampling frequency $= 2f_{max} = 2 \times 900$ Hz

or $f_s = 1800$ Hz.

Calculation of folding frequency:

Given levels of quantization $= L = 1024$ levels.

We have number of bits per sample $= b = \log_2 L$

Therefore, $b = \log_2 1024 \quad \therefore \; b = \frac{\log_{10} 1024}{\log_{10} 2}$

$\therefore$ $b = 10$ bits

Now, bit rate = samples/sec × bis/sample

Therefore,

$$\text{samples/sec} = \frac{\text{bit rate}}{\text{bits / sample}}$$

Here, bit rate = 10,000 bits/sec. (given)

and bits/sample $= b = 10$

Therefore

$$\text{samples/sec} = \frac{10,000}{10} = 1000 \text{ Hz.}$$

Now, aliasing starts after $\frac{f_s}{2}$, which is called as folding frequency.

$$\text{Folding frequency} = \frac{\text{samples / sec}}{2} = \frac{1000}{2}$$

$$\therefore \text{ Folding frequency} = \frac{f_2}{2} = 500 \text{ Hz}$$

(ii) $$\text{Nquist rate} = 2f_{max} = 2 \times 900 \text{ Hz}$$

$$\therefore \text{ Nyquist rate} = 1800 \text{ Hz}$$

(iii) Given signal is,

$$x_a(t) = 3\cos 600\pi t + 2\cos 1800\,\pi t \quad ...(i)$$

Resolution is given by,

$$\Delta = \frac{x_{max} - x_{min}}{L-1} \quad ...(ii)$$

Here, x_{max} will occur when cosine term in equation (i) is maximum that means equal to 1.

$$\therefore \quad x_{max} = 3 \times 1 + 2 \times 1 \text{ ... From equation (i)}$$

or $$x_{max} = 5$$

And x_{min} will occur when cosine term in equation (i) is minimum that means equal to -1.

Thus, $$x_{min} = 3(-1) + 2(-1) = -5$$

Given number of levels $= L = 1024$ levels.

Substituting these values in equation (i), we get

$$\Delta = \frac{5-(-5)}{1024-1} = \frac{10}{1023}$$

or $$\Delta = 9.775 \times 10^{-3} \quad \text{Ans.}$$

Example 6.18. Consider analog signal $x_a(t) = 100\cos(100\,\pi t)$. What is the discrete frequency $0 < F < \frac{f_s}{2}$ of a sinusoid that yields samples identical to those obtained with $f_s = 75$ Hz.

Solution : The given analog signal is

$$x_a(t) = 100\cos(100\pi t) \quad ...(i)$$

Comparing equation (i) with standard equation

$$x_a(t) = \cos(2\pi f t), \text{ we get,}$$

$$2\pi/T = 100\,\pi T$$

or $$f = 50 \text{ Hz}$$

Now, the discrete time signal $x(n)$ is obtained t by $\frac{n}{f_s}$ in equation (i).

Here f_s = sampling frequency = 75 Hz.

Therefore, $$x(n) = 100\cos\left(100\pi \cdot \frac{n}{f_s}\right) = 100\cos\left(\frac{100\pi n}{75}\right) \quad ...(ii)$$

Now, let us convert equation (ii) in the standard format, *i.e.*,

$$x(n) = 100\cos\left(2\pi \times \frac{50\,n}{75}\right) = 100\cos 2\pi\left(\frac{10}{15}\,n\right)$$

or $$x(n) = 100\cos\left(\frac{20\pi}{15}n\right)$$

or $$x(n) = 100\cos\left(2\pi - \frac{10\pi}{15}\right)n$$

But $\cos(2\pi - \theta) = \cos\theta$

Therefore, $$x(n) = 100\cos\left(\frac{-10\pi}{15}\right)n$$

and $\cos(-\theta) = \cos\theta$

$\therefore$ $$x(n) = 100\cos\left(\frac{10\pi}{15}\right)n$$

$\therefore$ $$x(n) = 100\cos 2\pi\left(\frac{5}{15}\right)n$$

or $$x(n) = 100\cos 2\pi\left(\frac{1}{3}\right)n \quad \text{...(iii)}$$

Now, the discrete time frequency is, $f = \frac{F}{f_s}$

or $$F = f \cdot f_s$$

Here $$f = \frac{1}{3} \quad \text{and } f_s = 75 \text{ Hz}$$

Therefore, $$F = \frac{1}{3} \times 75$$

or $$F = 25 \text{ Hz}$$

This signal frequency is within the range $0 < F < \frac{f_s}{2}$. Now, the sinusoidal which yields samples identical to those obtained by discretizing $x_a(t)$ is given by

$$y_a(t) = 100\cos 2\pi Ft$$

$\therefore$ $$y_a(t) = 100\cos 2\pi \times 25\, t$$

or $$y_a(t) = 100\cos 50\,\pi t$$ **Ans.**