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Subt Signal & System.
Class - B.tech 2nd yr (ECE)
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Topic: Properties of Z-transform

5. Differentiation in z-domain:

Statement: If $x(n) \xleftrightarrow{Z} X(Z)$,

then. $$n\,x(n) \xleftrightarrow{Z} -Z \frac{dX(Z)}{dZ}.$$

Proof: Acc. to definition of Z-transform

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad \text{--- (1)}$$

Differentiate both sides w.r.t. Z.

$$\frac{dX(Z)}{dZ} = \frac{d}{dZ}\left[\sum_{n=-\infty}^{\infty} x(n) Z^{-n}\right]$$

We can transfer $\frac{d}{dZ}$ inside the summation sign

$$\frac{dX(Z)}{dZ} = \sum_{n=-\infty}^{\infty} \frac{d}{dZ}\left[x(n)\, Z^{-n}\right]$$

$$\sum_{n=-\infty}^{\infty} (-n)\cdot x(n)\, Z^{-n-1}$$

But z^{-n-1} can be written as $z^{-n} \cdot z^{-1}$

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} (-n) \cdot x(n) \cdot z^{-n} z^{-1}$$

Since the limits of summations are in terms of n, we can take z^{-1} outside the summation

$$\therefore \frac{dX(z)}{dz} = z^{-1} \sum_{n=-\infty}^{\infty} (-n) \cdot x(n) z^{-n}$$

taking negative sign outside the summation

$$\frac{dX(z)}{dz} = -z^{-1} \sum_{n=-\infty}^{\infty} [n x(n)] z^{-n}$$

$$\therefore \frac{dX(z)}{dz} = -\frac{1}{z} \sum_{n=-\infty}^{\infty} [n x(n)] z^{-n}$$

$$\therefore -z \frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} [n x(n)] z^{-n} \quad \text{--- (2)}$$

comparing R.H.S. with eq (1) we can say that $\sum_{n=-\infty}^{\infty} [n x(n)] z^{-n}$ is the z-transform of $n x($

$$-z \frac{dX(z)}{dz} = z\{n x(n)\}$$

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Ex 1: Using differentiation property obtain z transform of unit Ramp sequence.

Sol: We know that unit Ramp sequence is given by

$$x(n) = n\,u(n) \quad \text{--- (1)}$$

Acc to differentiation Property we have:

$$Z\{n\,x(n)\} = -z\frac{dX(z)}{dz} \quad \text{--- (2)}$$

In this case we can write

$$Z\{n\,u(n)\} = -z\frac{d}{dz}[Z\{u(n)\}]$$

We have $Z\{u(n)\} = \frac{z}{z-1}$

$$\therefore Z\{n\,u(n)\} = -z\frac{d}{dz}\left[\frac{z}{z-1}\right]$$

$$\therefore Z\{n\,u(n)\} = -z\frac{d}{dz}\left[(z)\cdot(z-1)^{-1}\right]$$

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$$= -z\left[z \cdot \frac{d}{dz}\left[(z-1)^{-1}\right] + (z-1)^{-1}\frac{d}{dz}z\right]$$

$$= -z\left[z(-1)(z-1)^{-2} + (z-1)^{-1} \cdot 1\right]$$

$$= -z\left[\frac{-z}{(z-1)^2} + \frac{1}{(z-1)}\right]$$

$$= z\left[\frac{-z+z-1}{(z-1)^2}\right] = -z\left[\frac{-1}{(z-1)^2}\right]$$

$$\therefore Z\{n u(n)\} = \frac{z}{(z-1)^2}.$$

without using differentiation property.

Ex. 3.3.25 : Obtain the Z-transform of signal,

$x(n) = n\, a^n\, u(n)$

Soln. :

Let $x_1(n) = a^n u(n)$ $\therefore x(n) = n\, x_1(n)$

According to differentiation property,

$$n\, x_1(n) \overset{Z}{\longleftrightarrow} -Z \frac{d}{dZ} X_1(Z)$$

In this case we can write,

$$Z\{n\, a^n u(n)\} = -Z \frac{d}{dZ} \{a^n u(n)\}$$

Here the Z-transform of $a^n u(n)$ is given by,

$$Z\{(a^n u(n)\} = \frac{Z}{Z-a} \qquad \text{ROC} : |Z| > |a|$$

Putting this value in Equation (2) we get,

$$Z\{n\, a^n u(n)\} = -Z \frac{d}{dZ}\left[\frac{Z}{Z-a}\right]$$

$$\therefore Z\{n\, a^n u(n)\} = -Z \frac{d}{dZ}\left[Z \cdot (Z-a)^{-1}\right]$$

$$= -Z \left\{Z \cdot \frac{d}{dZ}\left[(Z-a)^{-1}\right] + (Z-a)^{-1} \cdot \frac{d}{dZ} Z\right\}$$

$$= -Z\left[Z \cdot (Z-a)^{-2} \cdot (-1) + (Z-a)^{-1} \cdot 1\right]$$

$$= -Z\left[\frac{-Z}{(Z-a)^2} + \frac{1}{(Z-a)}\right] = -Z\left[\frac{-Z+Z-a}{(Z-a)^2}\right]$$

$$= -Z\left[\frac{-a}{(Z-a)^2}\right]$$

$$\therefore \quad Z\{n a^n u(n)\} = \frac{aZ}{(Z-a)^2} \qquad \text{ROC} : |Z| > |a|$$

This is also standard Z-transform pair.

$$\boxed{\therefore\ n a^n u(n) \overset{Z}{\longleftrightarrow} \frac{aZ}{(Z-a)^2} \qquad \text{ROC} : |Z| > |a|}$$

Note : From this equation we can obtain Z-transform of unit ramp sequence, $n\,u(n)$. By putting $a = 1$ in Equation (3) we get,

$$Z\{n\,u(n)\} = \frac{Z}{(Z-1)^2} \qquad \text{ROC} : |Z| > |a|$$

Ex. 3.3.26 : The Z transform of DT signal $x(n)$ is given by $x(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z^2+4}$ ROC $|Z| > 2$. Determine Z transform and ROC of the following signals using properties of Z transform.

(i) $2^n x(n)$ (ii) $nx(n)$ (iii) $x(-n)$ (iv) $x(n-4)$.

Soln. :

(i) $2^n x(n)$

According to the scaling property,

$$a^n x(n) \overset{Z}{\leftrightarrow} X\left(\frac{Z}{a}\right)$$

Here $X(Z) = \frac{Z}{Z^2+4}$, ROC $|Z| > 2$

$$\therefore \quad 2^n x(n) \overset{Z}{\longleftrightarrow} \frac{Z/2}{\left(\frac{Z}{2}\right)^2 + 4} \qquad \text{ROC } \left|\frac{Z}{2}\right| > 2$$

$$\therefore \quad Z\{2^n x(n)\} = \frac{Z/2}{\frac{Z^2}{4} + 4}, \qquad \text{ROC} |Z| > 4$$

(ii) $n\,x(n)$

According to the differentiation property,

$$n\,x(n) \overset{Z}{\longleftrightarrow} -Z\frac{d}{dz}X(Z)$$

$$\therefore \; Z\{n\,x(n)\} = -Z\frac{d}{dz}\left\{\frac{Z}{Z^2+4}\right\} = -Z\{Z(-1)(Z^2+4)^{-2}+(Z^2+4)^{-1}\}$$

$$\therefore \; Z\{n\,x(n)\} = \frac{Z^2}{Z^2+4}-\frac{Z}{Z^2+4} = \frac{Z^2-Z}{Z^2+4}$$

The ROC remains same as X (Z).

(iii) $\mathbf{x(-n)}$

According to time reversal property,

$$x(-n) \overset{Z}{\longleftrightarrow} X(Z^{-1})$$

$$\therefore \quad Z\{x(-n)\} = \frac{Z^{-1}}{(Z^{-1})^2+4} \qquad \text{ROC } |Z^{-1}| > 2$$

$$= \frac{Z^{-1}}{Z^{-2}+4} \qquad \text{ROC } \left|\frac{1}{Z}\right| > 2 \text{ i.e. } 1 > 2|Z| \text{ i.e. } |Z| > \frac{1}{2}$$

$$\therefore \quad Z\{(x(-n)\} = \frac{Z}{1+4Z^2} \qquad \text{ROC } |Z| > \frac{1}{2}$$

(iv) $\mathbf{x(n-4)}$

According to time shifting property,

$$x(n-k) \overset{Z}{\longleftrightarrow} Z^{-k}X(Z)$$

$$\therefore \quad Z\{x(n-4)\} = Z^{-4}\left[\frac{Z}{Z^2+4}\right] = \frac{Z^{-3}}{Z^2+4}$$

Since k = 4 which is greater than zero, ROC is same as X (Z) except Z = 0.

3.3.8 Convolution of Two Sequences (Convolution Property) :

PTU - May 2009

Statement : If $x_1(n) \overset{Z}{\longleftrightarrow} X_1(Z)$ and $x_2(n) \overset{Z}{\longleftrightarrow} X_2(Z)$

Then $x_1(n) * x_2(n) \overset{Z}{\longleftrightarrow} X_1(Z) \cdot X_2(Z)$

and ROC is atleast the intersection of ROC of $X_1(Z)$ and $X_2(Z)$.

Proof : According to the definition of convolution, the convolution of $x_1(n)$ and $x_2(n)$ can be written as,

$$x(n) = x_1(n) * x_2(n) \qquad \ldots(3.3.15)$$

$$x(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \qquad \ldots(3.3.16)$$

Taking Z-transform of $x(n)$,

$$X(Z) = Z\left\{\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\right\} \qquad \ldots(3.3.17)$$

According to the definition of Z-transform we have,

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n} \qquad \ldots(3.3.18)$$

In Equation (3.3.17) take the bracket term as $x(n)$. Thus Equation (3.3.18) becomes,

$$\therefore Z\{x(n)\} = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\right] \cdot Z^{-n}$$

Rearranging the summation terms,

$$Z\{x(n)\} = \sum_{k=-\infty}^{\infty} x_1(k)\left[\sum_{n=-\infty}^{\infty} x_2(n-k)\, Z^{-n}\right] \qquad \ldots(3.3.19)$$

Now Z^{-n} can be written as,

$$Z^{-n} = Z^{-n} \times Z^{k} \times Z^{-k} = Z^{-(n-k)} \times Z^{-k}$$

Putting this value in Equation (3.3.19) we get,

$$Z\{x(n)\} = \sum_{k=-\infty}^{\infty} x_1(k) \left[\sum_{n=-\infty}^{\infty} x_2(n-k) Z^{-(n-k)} \cdot Z^{-k} \right]$$

In the second summation: the limits of summation are in terms of 'n'. So we can take Z^{-k} out of the summation sign.

$$\therefore \; Z\{x(n)\} = \left[\sum_{k=-\infty}^{\infty} x_1(k) \cdot Z^{-k} \right] \left[\sum_{n=-\infty}^{\infty} x_2(n-k) Z^{-(n-k)} \right]$$

In the second summation put $n - k = m$. The limits will change as follows :

when $n = -\infty \Rightarrow -\infty - k = m$ $\quad\therefore\; m = -\infty$

and when $n = +\infty \Rightarrow \infty - k = m$ $\quad\therefore\; m = \infty$

$$\therefore \; Z\{x(n)\} = \left[\sum_{k=-\infty}^{\infty} x_1(k) \times Z^{-k} \right] \left[\sum_{m=-\infty}^{\infty} x_2(m) Z^{-m} \right] \quad ...(3.3.20)$$

Compare each bracket of R.H.S. with the definition of Z– transform.

$$\therefore \; Z\{x(n)\} = X_1(Z) \cdot X_2(Z) \quad ...(3.3.21)$$

But here $\quad x(n) = x_1(n) * x_2(n)$

$$\therefore \; Z\{x_1(n) * x_2(n)\} = X_1(Z) \cdot X_2(Z)$$

$$\boxed{x_1(n) * x_2(n) \overset{Z}{\longleftrightarrow} X_1(Z) \cdot X_2(Z)}$$

ROC is atleast the intersection of $X_1(Z)$ and $X_2(Z)$.