TITLE OF E-CONTENT

MATRICES AND MATRIX OPERATIONS

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MATRICES AND MATRIX OPERATIONS

Learning Outcomes

In the present chapter you will learn about the following aspects;

- Understand the concept of matrices
- You will be able to apply matrix operations.
- Understand the properties of matrix.
- Rules of matrix multiplication
- Understand the concept of power of matrix
- Understand the concept of invertible matrix

INTRODUCTION

The subject of matrices had its origin in various types of problems. Of these, solutions of a given system of equations d liner transformations in geometry are extremely interesting. In 1857, the British mathematician Arthur Cayley formulated the general theory of matrices. He developed the properties of matrices as pure algebraic structure, though matrices as arrays of coefficients in homogeneous linear equation were recognized long before. A matrix is a very useful tool to analyze of various problems in different subjects.

DEFINITION OF MATRICES

A matrix is ordered set of numbers listed rectangular form;

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

OR

A matrices is simply a rectangular table of numbers written in either () or [] brackets. These symbols are also called the notation of matrix. Matrix has many application in science, engineering, computing and Economies. In economies, matrices is useful to study of stock market tends, optimization of profit, minimization of loss, input - output analysis etc. Do not confuse a matrix with determinants which use vertical bar, i.e.$\left|\ \right|$. Basically, matrix is a simple pattern of numbers on the other hand determinant gives us a single number. The size of matrix is written a_{ij} , where, i =row and j = columns. a_{ij} is the element of a matrix.

For examples

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow 2\times 2\ Matrix \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} \Rightarrow 4\times 1\ Matrix \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} 3\times 3 matrix$$

TYPES OF MATRIX

- **Square Matrix**: If a matrix has 'n' rows and 'n' columns then we say it is a square matrix. For example;

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2} \quad or \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

- **Diagonal Matrix**: It is a square matrix where all non-diagonal element is zero such that

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{12} & 0 \\ 0 & 0 & a_{13} \end{bmatrix}_{3\times 3}$$

- **Row Matrix**: A matrix with one row is called a row matrix.

$$A = \left[a_{11},\ a_{12},\ a_{13}\right]_{1\times 3}$$

- **Column Matrix**: A matrix with one column is called a column matrix.

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}_{3\times 1}$$

- **Zero Matrix**: If the all elements of an matrix is zero then it is called zero matrix.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- **Opposite Matrix**: If the all elements of an matrix multiply by negative sign then we get opposite matrix

$$A_{ij} = -A_{ij}$$

- **Transpose Matrix**: If we convert row to column and column to row of an matrix then we gent transpose of an matrix. For example;

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$$

Then;

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}_{3\times3}$$

It is also true for m × n matrix

We can also write;

$$A^T = A' = (a'_{ij}), \text{ where, } a'_{ij} = -a_{ij}$$

Example: Given;

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \textit{find } A^T$$

Solution:

$$A^T = A' = \begin{bmatrix} -1 & 2 \\ 0 & 3 \\ 2 & 1 \end{bmatrix}_{3x2}$$

RULES OF TRANSPOSITION

- $(A^1)^1 = A \text{..................}(i)$
- $(A+B)^1 = A^1 + B^1 \text{..................}(ii)$
- $(\alpha A^1) = \alpha A^1 \text{ and } (\alpha B)^1 = \alpha \beta^1 \text{.............}(iii) \quad (\alpha \text{ is constent})$
- $(AB)^1 = B^1 A^1 \text{........................}(iv)$

- **Symmetric Matrix**: A square matrix is said to be symmetric matrix, it is is equal to its transpose matrix.

$$i.e. \quad a_{ij} = a_{ji} \quad \forall \ i \ \& \ j$$
$$or \quad A = A' = A^T \qquad \{A = (a_{ij})n \, x \, n\}$$

- **Skew - symmetric Matrix**: It is defined as;

$$a_{ij} = -a_{ji}$$
$$or A = -A' = -A^T \{A = (a_{ij})_{n\times n} \ \& \ A = (a_{ji})_{n\times n}\}$$

- **Identity Matrix**: An identity matrix (I) is an diagonal matrix with all diagonal elements is equal to one. It is also known as unit matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \qquad I_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ .. & .. & .. & .. \\ 0 & 0 & 0 & ...1 \end{bmatrix}_{n\times n}$$

Or $\boxed{A \, I_n = I_n A = A}$

Example: Construct the 3 × 3 matrix $A = (a_{ij})_{3\times 3}$ with $a_{ij} = 2i - j$

Solution: Totalelement of matrix = 3 × 3 = 9

Given; $a_{ij} = 2i - j$

$$\therefore \quad a_{11} = 2x1 - 1 = 1, \ a_{12} = 2 - 2 = 0, \quad a_{13} = 2 - 3 = -1$$
$$a_{21} = 4 - 1 = 3, \quad a_{22} = 4 - 2 = 2, \quad a_{23} = 4 - 3 = 1$$
$$a_{32} = 6 - 2 = 4, \quad a_{33} = 6 - 3 = 2, \quad a_{23} = 4 - 3 = 3$$

Then Matrix;

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

Putting the values;

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 5 & 4 & 3 \end{bmatrix}_{3\times 3}$$

OPERATIONS OF MATRIX

Equality Matrix: Suppose, $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m \times n}$ be the two m×n matrices. Then A and B is said to be equal matrices if A = B

$$a_{ij} = b_{ij} \forall i = 1,2,------m \;\&\; j = 1,2,--------n$$

Thus, if both matrices have some dimension then they called equal. Otherwise they called unequal matrix such that $A \neq B$.

Example: Given;

$$\begin{pmatrix} 3 & y-1 \\ 2 & x \end{pmatrix} = \begin{pmatrix} z-2 & 2 \\ 4 & 3 \end{pmatrix}$$

Solve for x, y, and z.

Solution: Byequation of matrices

$$\begin{bmatrix} 3 & y-1 \\ 2 & x \end{bmatrix} = \begin{bmatrix} z-2 & 2 \\ 4 & 3 \end{bmatrix}$$

$$\therefore Z - 2 = 3 \text{ or } Z = 5, \; y - 1 = 2, \; y = 3 \text{ and } x = 3$$

Addition and Multiplication by a Scalar

Let $A = a_{ij})_{m \times n}$and $B = (b_{ij})$ be the two marticesthent the sum of A and B matrices is defined as;

$$A + B = (a_{ij})_{m \times n} + (b_{ij})_{m \times n}$$

$$A + B = (a_{ij} + b_{ij})_{m \times n}$$

If α is a real mumber, then

$$\alpha A = \alpha\, (a_{ij})_{m \times n} = (\alpha a_{ij})_{m \times n}$$

and $$\alpha B = \alpha\, (a_{ij})_{m \times n} = (\alpha b_{ij})_{m \times n}$$

$$\therefore \alpha A + \alpha B = (\alpha a_{ij} + \alpha b_{ij})_{m \times n}$$

Example : Given,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

Compute A + B, and $2A + \frac{1}{2}B$

Solution:

$$A + B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & -3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+1 & 2+0 & 3+2 \\ 4+0 & 2+0 & -3+1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & 5 \\ 4 & 4 & -2 \end{pmatrix}$$

And

$$2A + ½B = 2\begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & -3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 & 6 \\ 8 & 4 & -6 \end{pmatrix} + \begin{pmatrix} ½ & 0 & 1 \\ 0 & 1 & ½ \end{pmatrix}$$

$$= \begin{pmatrix} 2+½ & 4+0 & 6+1 \\ 8+0 & 4+1 & -6+½ \end{pmatrix} = \begin{pmatrix} 2½ & 4 & 7 \\ 8 & 5 & 5½ \end{pmatrix}$$

RULES OF MATRIX ADDITION AND MULTIPLICATION BY SCALARS

If A, B and C are $m \times n$ matrix and α and β are scalar then;

- $(A + B) + C = A + (B + C)$
- $A + B = B + A$
- $A + 0 = A$
- $A + (-A) = 0$
- $(\alpha + \beta) A = \alpha A + \beta A$
- $\alpha (A + B) = \alpha A + \alpha B$

MATRIX MULTIPLICATION

Let us consider two matrices A and B,such that

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} \quad and \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}_{3\times 3}$$

Then the product of A and B is denoted by AB and it is given by;

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}_{3\times3}$$

Example: Given;

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Compute AB and BA

Solution:

$$AB = \begin{bmatrix} 2x4 & + & 3x2 \\ 2x4 & + & 0x2 \\ 1x4 & + & 2x2 \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \\ 8 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4x2 + 2x3 & 4x2 + 2x0 & 4x1 + 2x2 \end{bmatrix} = \begin{pmatrix} 14 & 8 & 8 \end{pmatrix}$$

Problem : Given that A = $(a_{ij})_{m\times n}$ and B $(b_{ij})_{m\times p}$ then compute the product of both matrix

i.e. C = AB

Solution: The product of two matrix is given by;

$$C = AB$$

$$or \quad (cij)\ m\,x\,p \ = \ (aj)\ m\,x\,n\,(bij)\,n\,x\,p$$

$$or \quad Cij \quad = \ aij\ bij + a_{i2}b_{2j} + ... + a_{in}\ b_{nj}$$

$$\left\{\text{Product of } i^{\text{th}} \text{ row of A and } j^{\text{th}} \text{ column of B}\right\}$$

In general;

$$\begin{bmatrix} C_{11} & C_{12}--- & C_{1j}--- & C_{ip} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ C_{i1} & C_{i2}--- & C_{ij}--- & C_{ip} \\ . & . & . & . \\ . & . & . & . \\ C_{m2} & C_{m2}--- & C_{mj}--- & C_{mp} \end{bmatrix}_{m\times p} = \begin{bmatrix} a_{11} & a_{12}--- & a_{1k}--- & a_{in} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{i1} & a_{i2}--- & a_{ik}--- & a_{in} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{m1} & a_{m2}--- & a_{mk}--- & C_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} b_{11} & b_{12}--- & b_{1j}--- & b_{ip} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ b_{k1} & b_{k2}--- & b_{kj}--- & bk_{ip} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ b_{n1} & b_{n2}--- & b_{nj}--- & b_{np} \end{bmatrix}_{m\times p}$$

Properties or Rules of Matrix Multiplication

- Matrix Multiplication is not commutative:

  If A and B are two matrix, then

  $$AB \neq BA$$

  For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

  and, if AB and BA are both defined, then,

  $$\begin{aligned} AB &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1\times0+0\times1 & 1\times1+0\times0 \\ 0\times0+(-1)\times1 & 0\times1+(-1)\times0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{aligned}$$

And

$$BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0\times1+1\times0 & 0\times0+1\times(-1) \\ 1\times1+0\times0 & 1\times0+0\times(-1) \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

So, we can say; $AB \neq BA$

Hence commutative law does not held

Note: Sometime it may hold but it does not always hold

- Matrix Multiplication is Associative;

  If A, B and C are three $m\times n$, $n\times p$ and $p\times q$ matrices respectively

  Then, $(AB)C= A(BC)$

- Matrix Multiplication is distributive with respect to addition of matrices :

  Suppose A, B, C are three $m\times n$, $n\times p$ and $p\times q$ matrices respectively, then

  $$A(B + C) = AB + AC$$

- Matrix Multiplication by a unit Matrix:

  If a is square matrix of order $n\times n$ and I is the unit matrix of the same order, then

  $$AI = A = IA \ \{\text{Also, } I\,I = I\}$$

- If the product of two matrices is a zero matrix, then it is possible that none of them is a zero matrix, i.e.

  AB = 0, then A ≠ 0 and B ≠ 0

  Where RHS '0' is zero matrix

  Let

  $$A \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

  Then

  $$AB = \begin{bmatrix} 1\times 2 + (-1)\times 2 & 1\times 2 + (-1)\times 2 \\ -1\times 2 + 1\times 2 & -1\times 2 + 1\times 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, the product is zero matrix.

- Cancellation law does not hold in matrix multiplication:

  Suppose A, B, C are three matrices such that AB and AC are defined, then AB = AC does not imply B = C

  $$A = \begin{bmatrix} 3 & 4 \\ 2 & 7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 6 \\ 1 & 5 \end{bmatrix}$$

Then find AB, A^2, B^2, and $(A + B)^2$. Is $(A + B)^2 = A^2 + B^2 + 2AB$?

Solution: We have given

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 6 \\ 1 & 5 \end{bmatrix}$$

$$\therefore \quad A + B = \begin{bmatrix} 3+2 & 4+6 \\ 2+1 & 7+5 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 3 & 12 \end{bmatrix}$$

$$\text{and } AB = \begin{bmatrix} 3 & 4 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3\times 2 + 4\times 1 & 3\times 6 + 4\times 5 \\ 2x2 + 7x1 & 2\times 6 + 7\times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 38 \\ 11 & 47 \end{bmatrix}$$

Now

$$A^2 = \begin{bmatrix} 3 & 4 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 9+8 & 12+28 \\ 6+14 & 8+49 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 40 \\ 20 & 57 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 2 & 6 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 4+6 & 12+30 \\ 2+5 & 6+25 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 42 \\ 7 & 31 \end{bmatrix}$$

$$\therefore \quad A^2 + B^2 + 2AB = \begin{bmatrix} 17 & 40 \\ 20 & 57 \end{bmatrix} + \begin{bmatrix} 10 & 42 \\ 7 & 31 \end{bmatrix} + 2\begin{bmatrix} 10 & 38 \\ 11 & 47 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 40 \\ 20 & 57 \end{bmatrix} + \begin{bmatrix} 10 & 42 \\ 7 & 31 \end{bmatrix} + \begin{bmatrix} 20 & 76 \\ 22 & 94 \end{bmatrix}$$

$$= \begin{bmatrix} 17+10+20 & 40+42+76 \\ 20+7+22 & 57+31+94 \end{bmatrix}$$

$$= \begin{bmatrix} 47 & 158 \\ 49 & 182 \end{bmatrix} \qquad -------------- \ (1)$$

Also,

$$(A+B)^2 = \begin{bmatrix} 5 & 10 \\ 3 & 12 \end{bmatrix} \begin{bmatrix} 5 & 10 \\ 3 & 12 \end{bmatrix} = \begin{bmatrix} 5\times5+10\times3 & 5\times10+10\times12 \\ 3\times5+12\times3 & 3\times10+12\times12 \end{bmatrix}$$

$$= \begin{bmatrix} 55 & 170 \\ 51 & 174 \end{bmatrix} \text{---(ii)}$$

from (i) and (ii)

$$(A+B)^2 \neq A^2 + B^2 + 2AB$$

POWER OF MATRICES

Suppose A is a square matrix, then the power matrix is defined as; $A^2 = AA$

$A^3 = AAA$ and so on----------

In general $A^n = AAA......... A \ \{Here, \ A \ is \ repeated \ n \ times\}$

Example: Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ then prove that;

$$A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Solution: Given

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A.A = \begin{bmatrix} 1 & 1+1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1+2 \\ 0 & 1 \end{bmatrix}$$

Then $A^K = A^{K-1}.\, A = \begin{bmatrix} 1 & 1+(k-1) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad \text{Pr}\mathit{oved}$$

Idempotent Matrix: Let A be an square matrix then the product A by itself is called Idempotent matrix. It is defined as;

$AA = A$, $AAA = A^3 = A$

In General $A^n = A$

Orthogonal Matrix: Let A is the n×n square matrix then A is said to be orthogonal matrix if,

or $(A)_{n\times n} (A')_{n\times n} = I_{n\times n}$

Example: Given; $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}_{2\times 2}$

Prove that A is orthogonal iff $a^2 + b^2 = 1$

Solution: Given,

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}_{2\times 2}$$

Than $A' = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}_{2\times 2}$

Now by property of orthogonal matrix,

$$AA' = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + b^2 & ab - ba \\ ba - ab & b^2 + a^2 \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + b^2 & ab - ba \\ ba - ab & a^2 + b^2 \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{bmatrix}$$

Given, $a^2 + b^2 = 1$ then

$$AA' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \qquad \textit{Hence } \text{Proved}$$

SYSTEMS OF EQUATIONS IN MATRIX FORM

It is defined as;

2x + 3y = 4 --------- (1)

6x - y = 2 --------- (1)

Now, these equation can be written as;

$$A = \begin{bmatrix} 2 & 3 \\ 6 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad and \quad b = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \text{then;}$$

$$AX = \begin{bmatrix} 2 & 3 \\ 6 & -1 \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + 3y \\ 6x - y \end{bmatrix}$$

It is equivalent to the matrix equation

AX = b

Example: If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ then prove by mathematical induction

$$A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$$

Solution: Let $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2\times 2 & -4\times 2 \\ 1\times 2 & 1-2\times 2 \end{bmatrix}$$

$$A^3 = A.A = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix} \quad \textit{proved}$$

Example: (i) A matrix P is orthogonal if P'P = 1. Prove that if P is an n×n matrix whose columns are all of length 1 and mutually orthogonal then P is orthogonal.

(ii) Find out if A is an orthogonal matrix.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{bmatrix}$$

Solution: **(i)** Let $P = \left(P_1\ P_2\ ----\ P_n\right)$ *and* $P_i^{'}\ P_i = 1$

$$P'P = \begin{bmatrix} P_1^1 & P_1^1 P_2 & --- & P_1^1 P_n \\ P_2^1 P_1 & P_1^1 P_2 & --- & P_2^1 P_n \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ P_n P_1 & P_n^1 P_2 & ---- & P_n P_n \end{bmatrix} = \begin{bmatrix} 1\ 0 & --- & 0 \\ 0\ 1 & --- & 0 \\ . & --- & . \\ . & --- & . \\ 0\ 0 & --- & 1 \end{bmatrix} = I$$

$\therefore$ P is orthogonal

(ii) No, A is not orthogonal matrix because columns of A are not of length 1 then;

$$A'A \neq I_3$$

Example: **(i)** Let D be the 3×3 diagonal matrix with entries d_1, d_2 and along d_3 along the diagonal and zero's elsewhere. Let A = (aij) be an arbitrary 3×3 matrix. Compute AD and DA.Show that AD multiplies the i^{th} column of A be entry d_i while DA multiples the i^{th} row of A by entry d_i.

(ii) If D is the 3×3 diagonal matrix with entries $d_1 = 2$, d_2 and $d_3 = 4$, find the A such that AD = DA

Solution: **(i)** Given $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$AD = \begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} \\ d_1a_{21} & d_2a_{22} & d_3a_{23} \\ d_1a_{31} & d_2a_{32} & d_3a_{33} \end{bmatrix}_{3\times 3}$$ i^{th} column multiply by di ----- (i)

$$DA = \begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} \\ d_1a_{21} & d_2a_{22} & d_3a_{23} \\ d_1a_{31} & d_2a_{32} & d_3a_{33} \end{bmatrix}_{3\times 3}$$ i^{th} row multiply by di ----- (2)

By (1) & (2)

AD = DA

(ii) Given; $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}_{3\times 3}$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Example: For what value of β, $D = \begin{bmatrix} \beta & \beta^2 - 1 \\ 2 & \beta + 1 \end{bmatrix}$ is symmetric?

Solution: Given,

$$A = \begin{bmatrix} \beta & \beta^2 - 1 \\ 2 & \beta + 1 \end{bmatrix}$$

A is symmetric matrix then,

$$A = A^T = A^1$$

$$\therefore \begin{bmatrix} \beta & \beta^2 - 1 \\ 2 & \beta + 1 \end{bmatrix} = \begin{bmatrix} \beta & 2 \\ \beta^2 - 1 & \beta + 1 \end{bmatrix}$$

By equating the matrices,

$\beta^2 - 1 = 2$

$or\ \beta^2 = 3$

$or\ \beta = \pm\sqrt{3}$, the matrix is symmetric

1.6 Invertible matrix

A square matrix A is called invertible matrix, if

$AB = BA = I_n$

or $AB = I$

and $BA = I$ {Hence B is inverse of matrix A}

PROBLEM SET

(1) Given,

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} x & 1 \\ y & -1 \end{bmatrix}, \text{ find x and y}$$

(2) Let $B = \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix}, \ the \ prove \ B^n = \begin{bmatrix} 1 & nq \\ 0 & 1 \end{bmatrix}$

(3) If $A = \begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix}, \ find \ \ A^2 - 5A + 7I$

(4) If $A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 0 \\ 4 & 1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix} \ then \ prove \ that \ AB \neq BA$

(5) Let $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \ B = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}, \ prove \ that \ (AB)' = B'A'$

(6) Given,

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, \quad then \ prove \ A \ is \ idempatent \ matrix$$

(7) For $\mu = \dfrac{1}{\sqrt{2}}$ the following given matrix is orthogonal

$$A = \begin{bmatrix} \mu & 0 & \mu \\ \mu & 0 & -\mu \\ 0 & 1 & 0 \end{bmatrix}$$

(8) Show that $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \ \ is \, symmetic \ matrix$

(9) Give an example of three matrices X, Y and Z to show that if XY = ZX, then it does not necessary that y = z

(10) If α is a scalar, A and B are matrices of order 3 × 4, then show that α(A + B) = α A + αβ

Answers of the Problem Set

(1) x = 1 & y = 2

(3) $\begin{bmatrix} 11 & -6 \\ -15 & 23 \end{bmatrix}$

(9) $X = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}, \; y = \begin{bmatrix} -2 & 6 \\ 1 & -10 \\ 4 & 2 \end{bmatrix}, z = \begin{bmatrix} 4 & 2 \\ -4 & -6 \\ 3 & 2 \end{bmatrix}$

READINGS

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