CHAUDHARY CHARAN SINGH UNIVERSITY, MEERUT

TITLE OF E-CONTENT

INTEGRATION OF FUNCTIONS
Definite Integration and Its Economic Application

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INTEGRATION OF FUNCTIONS
Definite Integration and Its Economic Application

1.0 LEARNING OUTCOMES OF THE CHAPTER

After completion of the present chapter, you should be able to;

- Evaluate definite integrals and relationship between differentiation and integration
- Find the area between two curves by using definite integration.
- *Understanding economic application of definite integration*

THE DEFINITE INTEGRAL

Introduction

Let F(x) be a continuous function over the interval [a, b] and it has a derivative $f(x)$ i.e. $F'(x) = f(x) \forall x \in (a,b)$. Then the difference, F(b)-F(a), is called the definite integral of function $f(x)$ over the interval [a, b]. In the first section of the present chapter, this difference, F(b)-F(a), does not depend on indefinite integrals. On the other hand, definite integral of $f(x)$ depends only on the function $f(x)$ and its interval [a, b]. Definite integral can be written as;

$$\int_a^b f(x)dx = F(x)\Big|_a^b = \Big|F(x)\Big|_a^b = F(b) - F(a)$$

where, $F'(x) = f(x) \forall x \in (a,b)$ and the number 'b' and 'a' are the upper and lower limits respectively.

Steps of Evaluating Definite Integral

Let $I = \int_a^b f(x)dx$

- first, find the indefinite integral, $\int f(x)dx = F(x) + c$
- Substitute, x = b upper limit in this integral, i.e. F(b) +C
- Substitute, x = a lower limit in this integral i.e. F(a)+C
- Subtract, second {F(b)+c} from third {F(a)+C}

$$\therefore \int_a^b f(x)dx = F(x)\Big|_a^b = \Big|F(x)\Big|_a^b = F(b) - F(a)$$

Example 1: Find, $\int_a^b x \ dx$

Solution: Let $I = \int_a^b x \ dx$

$$I = \left| \frac{x^2}{2} + c \right|_a^b$$

$$= \left[\frac{b^2}{2} + c \right] - \left[\frac{a^2}{2} + c \right]$$

$$= \left[\frac{b^2}{2} - \frac{a^2}{2} \right] = \frac{1}{2}(b^2 - a^2)$$

- **Some Basic Properties of Definite Integral**

- $\int_a^b F(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_b^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \int_{c_2}^b f(x)dx \quad \{c_1, c_2 \in [a,b]\}$
- $\int_a^a f(x)dx = 0 \qquad \{F(a) - F(a) = 0\}$
- $\int_a^b f(x)dx = \int_a^b f(y)dy = \int_a^b f(z)dz$
- $\int_0^a f(x)dx = \int_0^a f(x-a)dx$
- $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$
- $\frac{d}{dt}\int_{b(t)}^{a(t)} f(x)dx = f'(t) = f\{b(t)\}.b'(t) - f\{a(t)\}.a'(t)$
  - Every continuous function is integrable, if this function has an anti-derivative i.e.

$$\boxed{F'(x) = f(x), \ \forall x \in (a,b)}$$

Example 2: Find $\int_0^1 (2x + \frac{1}{x})dx$

Solution: Let $I = \int_1^2 (2x + \frac{1}{x})dx$

$$= \left| \frac{2x^2}{2} + \log x \right|_1^2$$

$$= \left| x^2 + \log x \right|_1^2$$

= [4 + log2]-[1+0]

= 3 + log 2

Example 3: Find the area of the parabola $x^2 = 4$ by between $x - axis$ and its ordinate at $x = 3$

Solution: The required area = $\int_0^3 ydx$

$$= \int_0^3 \frac{x^2}{4b}dx \qquad \left\{\because y = \frac{x^2}{4b}\right\}$$

$$= \frac{1}{4b}\left[\frac{x^3}{3}\right]_0^3$$

$$= \frac{1}{4b}\left[\frac{27}{3} - 0\right] = \frac{9}{4b}$$

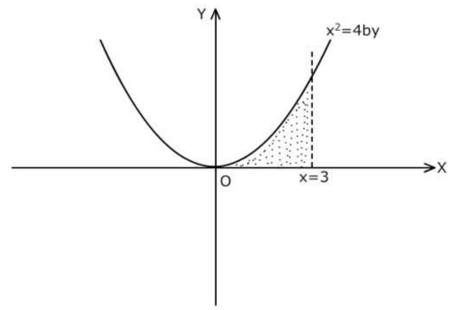


Figure 5

Example 4: find $\int_1^4 |x-2|dx$

Solution: Let

$$|x-2| = \begin{cases} x-2 & If\ x \geq 2 \\ -(x-2) & If\ x < 2 \end{cases}$$

Then $\int_1^4 |\text{x-2}|\, dx = \int_1^2 -(x-2)dx + \int_2^4 (x-2)dx$ {By property of Integration}

$$= \left[\frac{-x^2}{2} + 2x\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$$

$$= \left[\left(-\frac{4}{2}+4\right) - \left(-\frac{1}{2}+2\right)\right] + \left[\left(\frac{16}{2}-8\right) - \left(\frac{4}{2}-4\right)\right]$$

$$= 2 - \frac{3}{2} + 0 + 2. = \frac{5}{2}$$

Example 5: Find the area between the regions of parabola $y = x^2$ and straight line $y = |x|$ over the interval [-1,1]or $\{(x,y)x^2 \leq y \leq |x|\}$

Solution: Given $y = x^2$ and $y = |x|$ i.e. $y = x$ or $y = -x$

The required Area

= Area OAB + Area OCD

= 2 *Area OAB

(Because, curve is symmetrical about the y axis)

$$= 2\left[\int_0^1 xdx - \int_0^1 x^2 dx\right]$$

$$= 2\left[\left|\frac{x^2}{2}\right|_0^1 - \left|\frac{x^3}{3}\right|_0^1\right]$$

$$= \quad 2\left[\left(\frac{1}{2}-0\right) - \left(\frac{1}{3}-0\right)\right] = \quad 2/3$$

square units

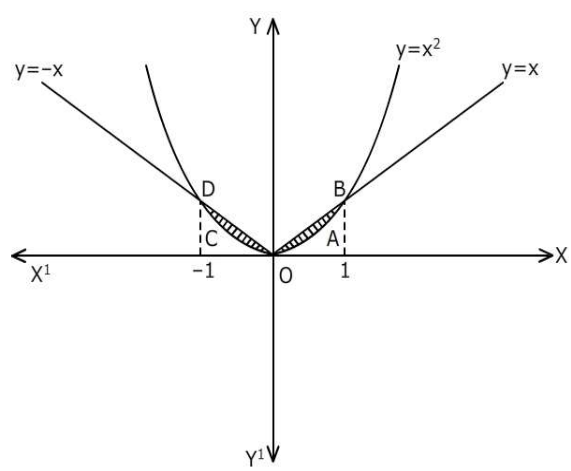


Figure 6

Example 6: Evaluate $\int_0^T \left(\frac{K}{T}\right) e^{-Qt} dt, where$ T> 0 and K and Q are positive constants.

Solution: Let W(T) = $\int_O^T \left(\frac{K}{T}\right) e^{-Qt} dt$

$$= \frac{K}{T}\int_O^T e^{-Qt} dt$$

$$= \frac{K}{T}\left[\frac{-e^{-Qt}}{Q}\right]_O^T$$

$$= \frac{K}{TQ}\left[(-e^{-QT})-(-e^{\circ})\right]$$

$$W(T) = \frac{K}{TQ}\left[1-e^{-QT}\right]$$

Example 7: Find the area included between the two parabola i.e. $y^2 = 4x$ and $x^2 = 4y$

Solution: Given, $y^2 = 4x \; \& \; x^2 = 4y$

Solving both, we get;

$$\left(\frac{x^2}{4}\right) = 4x$$

Or, $x(x^3 - 64) = 0$

So, $x = 0 \; \& \; 4$

The required area = Area OBCD

$$= \int_O^4 \left(\sqrt{4x} - \frac{x^2}{4}\right) dx$$

$\left\{\because y^2 = 4x \; \& \; y = x^2/4\right\}$

$$= 2\left[\frac{x^{3/2}}{3/2} - \frac{x^3}{12}\right]_0^4$$

= 5.3 square unit.

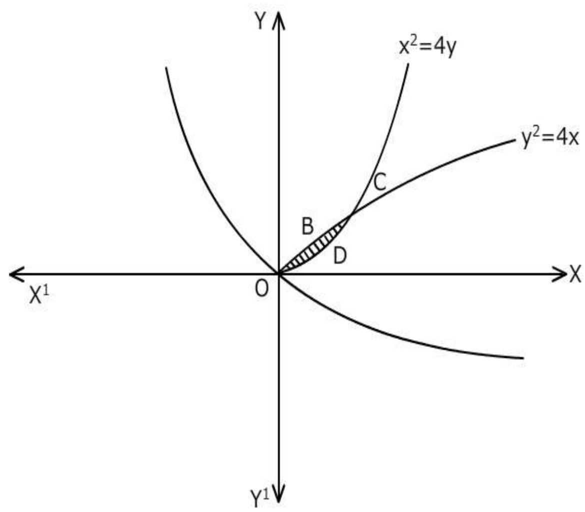


Figure 7

Example 8: Find $\frac{d}{dx}\int_{x}^{x^2} e^{-4^2} du$

Solution: By the direct property of integration, we get;

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x)dx$$

$$= f\{(b(x)\}b'(x) - f\{a(x)\}a'(x)$$

Then,

$$\frac{d}{dx}\int_{x}^{x^2} e^{4^2} du$$

$$= e^{-(x^2)^2} - 2x - e^{-x^2}.1$$
$$= e^{-x^2}\{2xe^{-x^2} - 1\}$$

ECONOMIC APPLICATION OF DEFINITE INTEGRATION

Introduction

Integration has an important role in economics. The present section shows the role of integration in economics by illustrating some important examples.

Important Results of Integration in Economics

- If $f(r)$ is the function of individuals income over the interval [a, b], then the no. of individuals with incomes in [a, b]

$$= n\int_a^b f(r)dr$$

- Total income of individuals = $n\int_a^b rf(r)dr$ $\{r \Rightarrow earning\}$

- The mean income of the individuals is given by

$$m = \frac{\int_a^b r\,f(r)dr}{\int_a^b f(r)dr}$$

Example 9: If the income distribution of population over interval [a, b] is given by, $f(r) = Ar^{-5/2}$ {A is a positive constant}, then determine mean income in the given group.

Solution: Let $\int_a^b f(r)dr = \int_a^b Ar^{-5/2}dr = A\left[-\frac{2}{3}r^{-3/2}\right]_a^b = \frac{2}{3}A\left(\left|a^{-3/2} - b^{-3/2}\right)$

And $\int_a^b rf(r)dr = \int_a^b Ar.r^{-5/2}dr$

$= \text{A} \int_a^b r^{-3/2}dr = 2A\left[a^{-1/2} - b^{-1/2}\right]$

So, the mean income of the group is given by

$$\text{m}= \frac{2A(a^{-1/2} - b^{-1/2})}{2/3A(a^{-3/2} - b^{-3/2})} = 3\frac{(a^{-1/2} - b^{-1/2})}{(a^{-3/2} - b^{-3/2})}$$

Now, suppose b is very large then $b^{-1/2}$ and $b^{-3/2}$ close to zero, then m≈3a

Then, the mean income of the group is 3a.

Economic Application of Integration

There are several other economic applications of integration. Some results are given below;

- **Consumer surplus (CS) and producer surplus (PS)**: These can be also calculated by using definite integral. Consumer surplus is given by;

$$CS = \int_o^x f(x)dx - p \times x$$

Here, $f(x) \Rightarrow$ demand of x commodity, P⇒ Price of x commodity

And, producer surplus is given by,

$$PS = x \times p - \int_o^x f(x)dx$$

- **The present discounted value is given by**;

$$\text{PDV} = \int_o^T f(t)e^{-rt}dt$$

- **The future discounted value is given by**;

$$\text{FDV} = \int_o^T f(t)e^{r(T-t)}dt$$

- **The discounted value at time is given by**;

$$\text{DV} = \int_{t=S}^{T} f(t)e^{-r(t-s)}dt$$

Example 10: If marginal revenue (MR) = $16-q^2$, find the maximum total revenue, also find the total, average revenue demand.

Solution: When TR is maximum, then MR= 0

$$\therefore 16-q^2=0 \Rightarrow q=\pm 4$$

$$\therefore TR=\int_0^4 MR\ \text{dq}=\int_0^4 (16-q^2)dq=\left[16q-\frac{q^3}{3}\right]_0^4=\frac{128}{3}$$

Total Revenue (TR) = $\int (16-x^2)dx = 16x-\frac{x^3}{3}+c$ when $x=0$ then $c=0$

$$\text{Average Revenue (AR)} = \frac{TR}{q}=16-\frac{q^2}{3}$$

Then, Demand (AR) =P = $16-q^2/3$

Example 11: If marginal propensity to consume (MPC) function is given as follows; $\frac{dc}{dy}=0.5-0.001y$, then find total consumption function. Given at income zero, c is 0.02.

Solution: $\therefore C=\int \frac{dc}{dy}.dy=\int (0.5-0.001y=0.5y-\frac{0.001}{2}y^2+A$

At $\therefore y=0$, then, C = 0.2, Hence, A = 0.2

$$\therefore C=0.5y-0.0005y^2+0.2$$

Example 12: The sales of a product is depicted by a function S(t) = $100e^{-0.5t}$, where t is number of years since the launching of the product, find

a) The total sales in the first three years
b) The sales in forth year &
c) The total sales in the future

Solution: a) $S(3)=\int_0^3 100e^{0.5t}dt=155.40$

b) $S(4)-S(3),$

$$S_4 = \int_3^4 100e^{0.5t}dt = 17.6$$

c) $$S(\infty) = \int_0^\infty 100\, e^{0.5t}dt = 200$$

Example 13: If the demand function is; $P = 30 - 2x - x^2$ and the demand is 3, what will be the consumer surplus (CS)?

Solution: Given, $P = 30 - 2x - x^2$

For $x = 3,$ then p = 20

$\therefore$ CS = $\int_0^3 (30 - 2x - x^2)dx - P \times x$

$$= \left[30x - \frac{2x^2}{2} - \frac{x^3}{3} \right]_0^3 - 3 \times 20$$

= 90-9-9-60=12 units

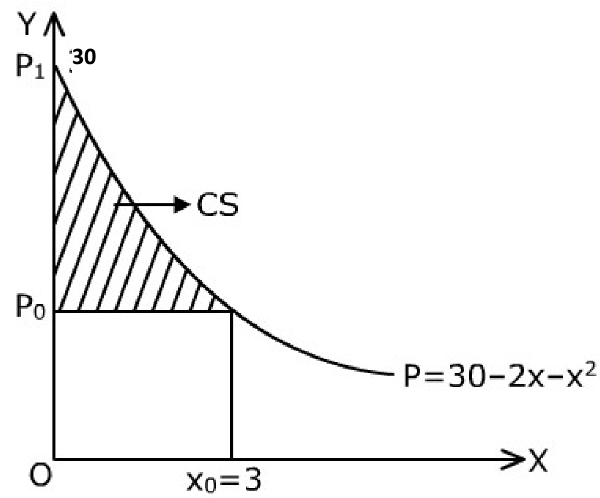


Figure 8

Example 14: The demand and supply laws are $P_d = (6 - x)^2$ and P_s=14+x respectively. Find the consumer surplus (CS), If;

(i) The demand and price are determined under perfect competition and;

(ii) The demand and price are determined under monopoly and the supply function is identified with marginal cost function.

Solution: (i) CS under perfect competition: at the equi8librium

$$(6 - x)^2 = 14 + x \Rightarrow x = 2$$

Then, P=14+x=16

$$\therefore CS = \int_0^2 (36 - 12x + x^2)dx - 16 \times 2 = 56/3$$

(ii) CS under monopoly;

$$TR = P_d x = (36 - 12x + x^2)x = 36x - 12x^2 + x^3$$

$$\therefore MR = 36 - 24x + 3x^2$$

And supply price: $P_s = 14 + x,$ supply function P_s=MC

To maximization of profit we know that,

MR=MC

$$36-24x+3x^2=14+x$$

i.e. $x=1, or, 7.33$

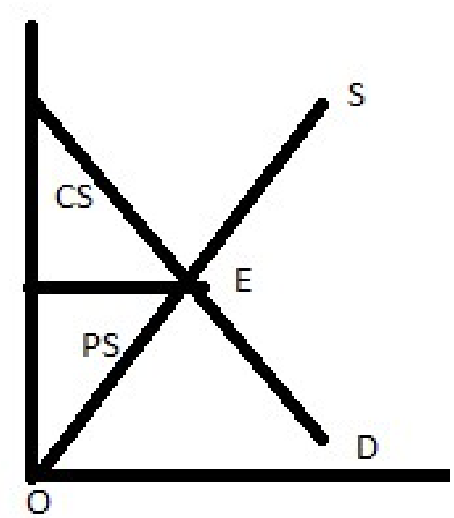
At $x=1,$ then, P_d=25

∵ Hence, $CS=\int_0^1(36-12x+x^2)dx-25x)=\frac{16}{3}$ unit

Similarly, we obtain CS at $x=7.33$

Example 15: Obtain the producer surplus, when the demand and supply function is given;

$D=20-4x$ and $S=4+4x$



Solution: At equilibrium condition,

Demand(D) = Supply (S)

$$20-4x=4+4x$$

$$or, 8x=16$$
$$then; x=2$$
$$and, P=4+8=12$$

And, P=4+8=12

Then, producer surplus (PS)

$$=P\times x-\int_0^2(4+4x)dx=24-[4x+2x^2]_0^2$$
$$=24-16=8units$$

PROBLEM SET

1. Find the definite integral for the following:

   (i) $\int_{-1}^{1} e^x \, dx$ (ii) $\int_0^2 \left(t^3 - t^2\right)dt$ (iii) $\int_1^3 \frac{3y}{10} dy$

2. Find, (i) $\frac{d}{dx}\int_0^x t^2 dt$ (vi) $\frac{d}{du}\int_{-u}^{u} e^{-v^2} dv$ (iii) $\frac{d}{du}\int_{-u}^{u} \frac{1}{\sqrt{x^4+1}} dx$

3. Find the area of line $y = 4x$ between $x - axis$ and the ordinate $x = 4$

4. Find the area intercepted between the line $3x + 2y = -12$ and the parable $y = \frac{3}{4}x^2$

5. Find the area between the parabolas; $y^2 = 4ax$ and $x^2 = 4ay, a > 0$

6. Prove that

   $$\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \text{ If } f(2a - x) = f(x)$$

   $$= 0 \qquad \text{If } f(2a - x) = -f(x)$$

7. Evaluate

   (i) $\int_0^1 (t + \sqrt{t} + \sqrt[4]{t})dt$ (ii) $I = \frac{1}{2000}\int_{1000}^{3000} f(t)dt$

   Given $F(t) = 4000 - t - \frac{3000000}{t}$

8. Prove that $F(t^*) = \frac{1}{b-a}\int_a^b f(t)dt$

   If $f(t)$ is continuous function over the interval [a,b] and $t^* \in (a,b)$
   $\left\{ Hint : Put\ F(t) = \int_a^t f(x)dx \right\}$

9. If the inverse demand function of commodity Q is given; $P = 3q^{-1/2}$ and presently 100 units are being sold, then find the consumer surplus.

10. Let interest rate will vary and represent by r(t). What is the present value of a flow of income P(t) from t=a to t=b using this variable interest rate?

ANSWERS OF PROBLEM SET

1. (i) $\frac{e^2-1}{e}$ (ii) $\frac{4}{3}$ (iii) $\frac{39}{10}$

2. (i) x^2 (ii) $2e^{-u^2}$ (iii) $\frac{1}{2\sqrt{u^4+1}}$

3. 32 sq. units

4. 27 sq. units

5. $\frac{16}{3}a^3$ sq. units

7. (i) $\frac{13}{12}$ (ii) $I \approx 352$

9. 30 units

10. $Ans. \int_a^b e^{-\int_a^t r(s)ds} P(t)dt$

Additional Readings

- Allen, R.G.D, *Mathematical Analysis for Economists*, London: Macmillan and Co. Ltd
- Chiang, Alpha C., *Fundamental Methods of Mathematical Economics*, New York: McGraw Hill
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