FARADAY'S LAW AND DISPLACEMENT CURRENT

Two topics will be discussed :

(i) Faraday's Law – about the existence of electromotive force (emf) in the magnetic field

(ii) Displacement current – that exists due to time varying field

That will cause the modification of Maxwell's equations (in point form - static case) studied previously and hence becomes a concept basic to the understanding of all fields in electrical engineering.

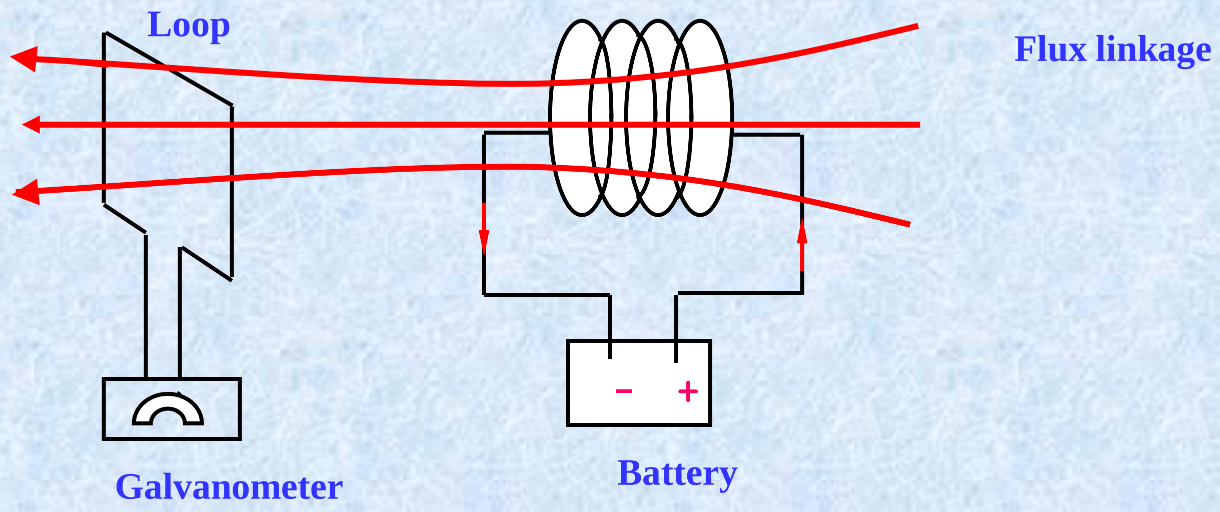
9.1 FARADAY'S LAW

Michael Faraday – proved that if the current can produce magnetic field, the reverse also will be true.

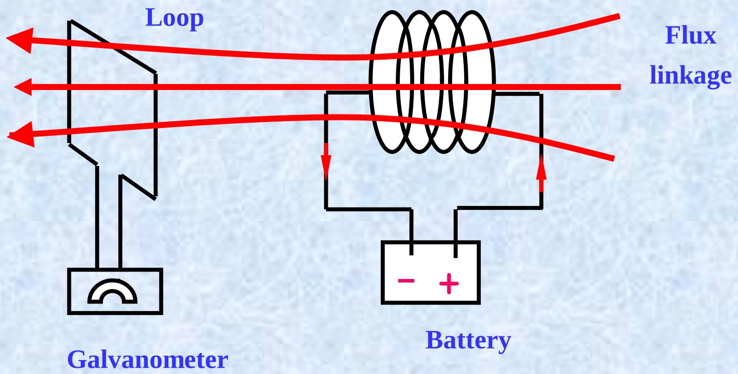
Proven only after 10 years in 1831.

The magnetic field can produce current in a loop, only if the magnetic flux linkage the surface of the loop is time varying.

Faraday's Experiment :



- Current produced magnetic field and the magnetic flux is given by :



$$\psi_m = \int_s \overline{B} \bullet d\overline{s} \qquad (1)$$

- No movement in galvanometer means that the flux is constant.

- Once the battery is put off – there is a movement in the galvanometer needle.

- The same thing will happen once the battery is put on - but this time the movement of the needle is in the opposite direction.

Conclusions : The current was induced in the loop

- when the flux varies

- once the battery is connected

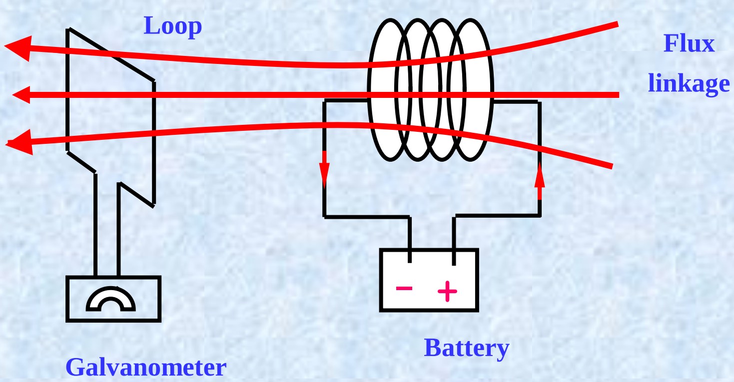
- if the loop is moving or rotating

Induced current will induced electromotive voltage or induced emf V_{emf} given by :

$$V_{emf} = -N\frac{\partial \psi}{\partial t} = -N\frac{\partial}{\partial t}\int_s \overline{B}\cdot d\overline{s} \quad (V) \qquad (2)$$

where N = number of turns

Equation (2) is called Faraday's Law



Lenz's Law summarizes the –ve sign is that : The induced voltage established opposes the the flux produced by the loop.

In general, Faraday's law manifests that the V_{emf} can be established in these 3 conditions :

- **Time varying field – stationary circuit (Transformer emf)**
- **Moving circuit – static field (Motional emf)**
- **Time varying field - Moving circuit (both transformer emf and motional emf exist)**

TIME VARYING FIELD – STATIONARY CIRCUIT

(TRANSFORMER EMF)

$$V_{emf} = -N\int_s \frac{\partial \overline{B}}{\partial t}\cdot d\overline{s} \quad (V)$$

(3)

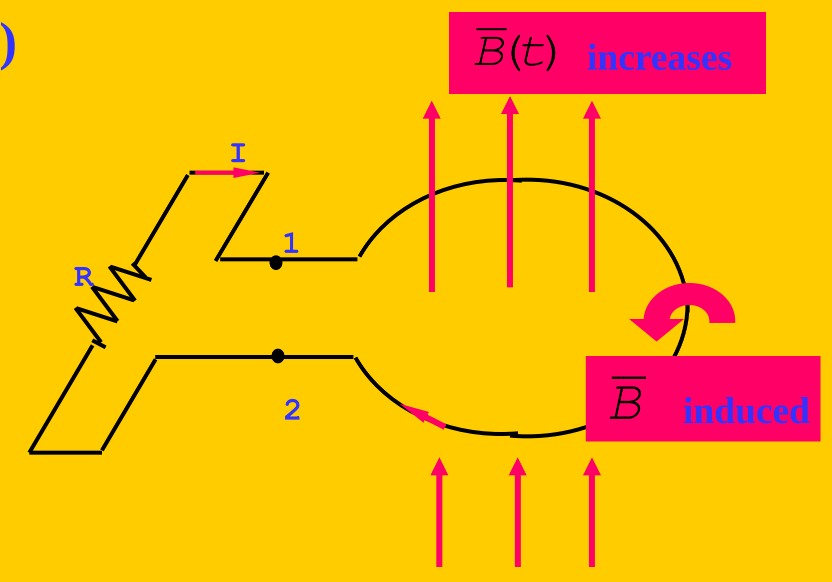


Diagram shows a circle loop with the surface area S placed in the magnetic field $\overline{B}(t)$

V_{emf} = the potential difference at terminal 1 and 2.

From electric field :

$$V_{emf} = \oint_\ell \overline{E} \bullet d\overline{\ell}$$

(4)

If N = 1 :

$$V_{emf} = \oint_\ell \overline{E} \bullet d\overline{\ell} = -\int_s \frac{\partial \overline{B}}{\partial t} \bullet d\overline{s}$$

(5)

$$V_{emf} = \oint_{\ell} \overline{E} \bullet d\overline{\ell} = -\int_{s} \frac{\partial \overline{B}}{\partial t} \bullet d\overline{s}$$

Using Stoke's theorem :

$$\int_{s} \left(\nabla \times \overline{E}\right) \bullet d\overline{s} = -\int_{s} \frac{\partial \overline{B}}{\partial t} \bullet d\overline{s} \qquad (6)$$

Hence Maxwell's equation becomes :

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \qquad (7)$$

MOVING CIRCUIT – STATIC FIELD (MOTIONAL EMF)

Force :

$$\overline{F}_m = q\left(\overline{u} \times \overline{B}\right)$$

$$\overline{E}_m = \frac{\overline{F}_m}{q} = \left(\overline{u} \times \overline{B}\right)$$

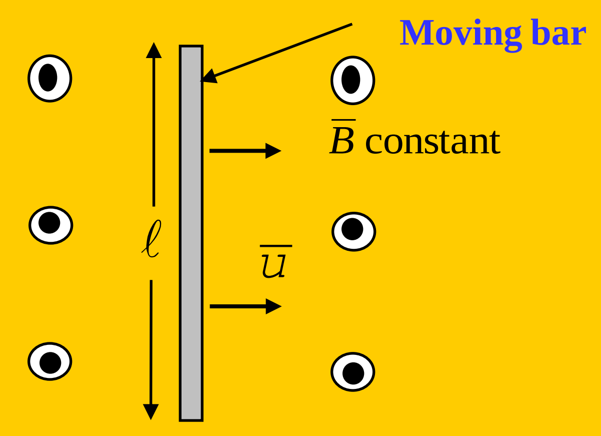


Diagram shows a bar moving with a velocity $\overline{u}$ in a static field $\overline{B}$.

Hence :

$$V_{emf} = \oint \overline{E}_m \bullet d\overline{\ell} = \oint \left(\overline{u} \times \overline{B}\right) \bullet d\overline{\ell}$$

Fleming's Right hand rule

Thumb – Motion

1st finger – Field

Second finger - Current

9.1.3 TIME VARYING FIELD - MOVING CIRCUIT

Both transformer emf and motional emf exist

$$V_{emf} = \oint \overline{E} \bullet d\overline{\ell} = \int_s -\frac{\partial \overline{B}}{\partial t} \bullet d\overline{s} + \oint \left(\overline{u} \times \overline{B} \right) \bullet d\overline{\ell}$$

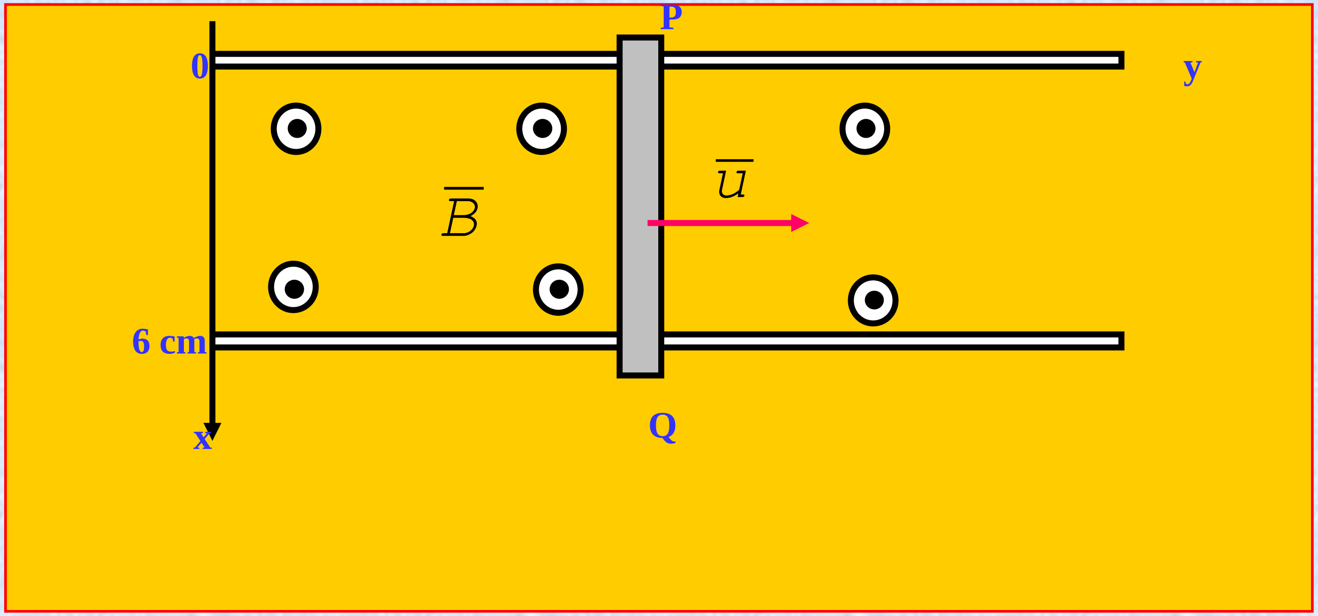
A conducting bar moving on the rail is shown in the diagram. Find an induced voltage on the bar if :

(i) A bar position at y = 8 cm and $\bar{B} = 4\cos 10^6 t\,\hat{z}\; mWb/m^2$

(ii) A bar moving with a velocity $\bar{u} = 20\hat{y}\; m/s$ **and** $\bar{B} = 4\hat{z}\; mWb/m^2$

(iii) A bar moving with a velocity $\bar{u} = 20\hat{y}\; m/s$ **and**

$\bar{B} = 4\cos(10^6 t - y)\,\hat{z}\; mWb/m^2$



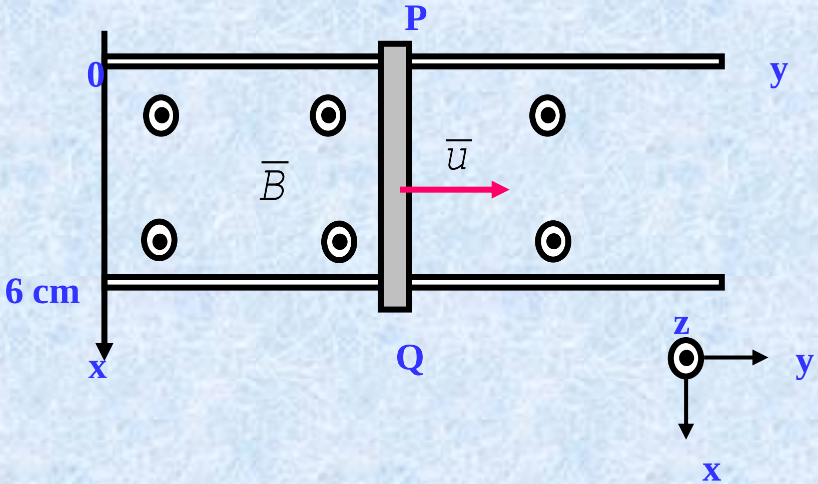
Solution :

(i) Transformer case :

$$V_{emf} = -\int \frac{\partial \overline{B}}{\partial t} \bullet d\overline{s}$$

$$= \int_{y=0}^{0.08} \int_{x=0}^{0.06} 4(10^{-3})(10^{6}) \sin 10^{6} t \, dxdy$$

$$= 19.2 \sin 10^{6} t \ (V)$$



$$\overline{B} = 4 \cos 10^{6} t \ \hat{z} \ \ mWb / m^{2}$$

According to Lenz's law when $\overline{B}$ increases point P will be at the higher potential with respect to point Q. (B induced will oppose the increasing $\overline{B}$)

(ii) Motional case :

$$V_{emf} = \int (\overline{u} \times \overline{B}) \bullet d\overline{\ell}$$

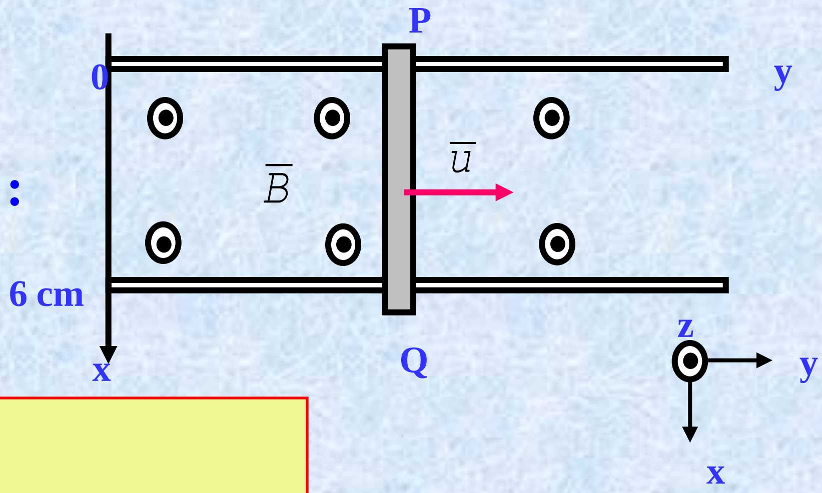
$$= \int_{x=0.06}^{0} (20\hat{y} \times 4\hat{z}) \bullet dx \, \hat{x}$$

$$= -4.8 \, mV$$

Remember : the direction of $d\overline{\ell}$ is opposed the current induced in the loop.

(iii) Both transformer and motional case :

$$\overline{B} = 4\cos(10^6 t - y)\,\hat{z}\ mWb/m^2$$



$$V_{emf} = -\int \frac{\partial \overline{B}}{\partial t} \bullet d\overline{s} + \int (\overline{u} \times \overline{B}) \bullet d\overline{\ell}$$

$$= \int_{x=0}^{0.06} \int_{0}^{y} 4(10^{-3})(10^6)\sin(10^6 t - y')\,dy'\,dx$$

$$+ \int_{0.06}^{0} \left[20\hat{y} \times 4(10^{-3})\cos(10^6 t - y)\hat{z}\right] \bullet dx\,\hat{x}$$

$$= 240\cos(10^6 t - y')\Big|_0^y - 80(10^{-3})(0.06)\cos(10^6 t - y)$$

$$= 240\cos(10^6 t - y) - 240\cos 10^6 t - 4.8(10^{-3})\cos(10^6 t - y)$$

$$\approx 240\cos(10^6 t - y) - 240\cos 10^6 t$$

$$V_{emf} = -\int \frac{\partial \overline{B}}{\partial t} \bullet d\overline{s} + \int (\overline{u} \times \overline{B}) \bullet d\overline{\ell}$$

$$\approx 240\cos(10^6 t - y) - 240\cos 10^6 t$$

1. Displacement Electric Current

The displacement current is neither the conduction current nor the convection current, which are formed by the motion of electric charges. It is a concept given by J. C. Maxwell.

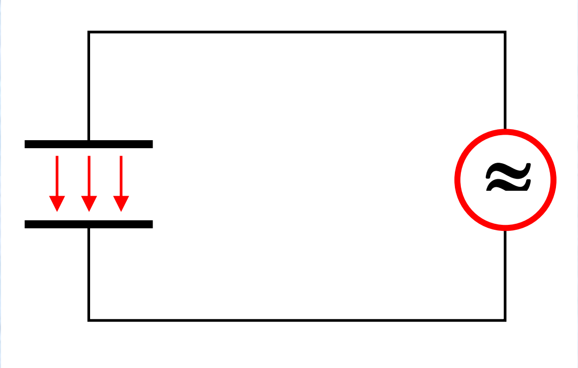
Based on the principle of electric charge conservation, we have

$$\oint_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} = -\frac{\partial q}{\partial t} \qquad \nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}$$

For static fields $\oint_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} = 0 \qquad \nabla \cdot \boldsymbol{J} = 0$

which are called the continuity equations for electric current.

For time-varying electromagnetic fields, since the charges are changing with time, the electric current continuity principle cannot be derived from static considerations. Nevertheless, an electric current is always continuous. Hence an extension of earliest concepts for steady current need to be developed.



The current in a vacuum capacitor is neither the conduction current nor the convection current, but it is actually the displacement electric current.

Gauss' law for electrostatic fields, $\oint_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = q$, is still valid for time-varying electric fields, we obtain

$$\oint_S \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) \cdot \mathrm{d}S = 0 \quad \Longrightarrow \quad \nabla \cdot \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) = 0$$

Obviously, the dimension of $\frac{\partial \boldsymbol{D}}{\partial t}$ is the same as that of the current density.

British scientist, James Clerk Maxwell named $\frac{\partial \boldsymbol{D}}{\partial t}$ the density of the displacement current, denoted as $\boldsymbol{J}_{\mathrm{d}}$, so that

$$\boldsymbol{J}_{\mathrm{d}} = \frac{\partial \boldsymbol{D}}{\partial t}$$

We obtain $\oint_S (\boldsymbol{J} + \boldsymbol{J}_{\mathrm{d}}) \cdot \mathrm{d}\boldsymbol{S} = 0$ $\qquad \nabla \cdot (\boldsymbol{J} + \boldsymbol{J}_{\mathrm{d}}) = 0$

The introduction of the displacement current makes the time-varying total current continuous, and the above equations are called the principle of total current continuity.

The density of the displacement current is the time rate of change of the electric flux density, hence

For electrostatic fields, $\frac{\partial \boldsymbol{D}}{\partial t} = 0$, and the displacement current is zero.

In time-varying electric fields, the displacement current is larger if the electric field is changing more rapidly.

In imperfect dielectrics, $\boldsymbol{J}_{\mathrm{d}} >> \boldsymbol{J}_{\mathrm{c}}$, while in a good conductor, $\boldsymbol{J}_{\mathrm{d}} << \boldsymbol{J}_{\mathrm{c}}$.

Maxwell considered that the displacement current must also produce magnetic fields, and it should be included in the Ampere circuital law, so that

$$\oint_l \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \int_S (\boldsymbol{J} + \boldsymbol{J}_\mathrm{d}) \cdot \mathrm{d}\boldsymbol{S}$$

i.e.

$$\oint_l \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \int_S (\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}) \cdot \mathrm{d}\boldsymbol{S} \qquad \nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

Which are Ampere's circuital law with the displacement current. It shows that a time-varying magnetic field is produced by the conduction current, the convection current, and the displacement current.

The displacement current, which results from time-varying electric field, produces a time-varying magnetic field.

The law of electromagnetic induction shows that a time-varying magnetic field can produce a time-varying electric field.

Maxwell deduced the coexistence of a time-varying electric field and a time-varying magnetic field, and they result in an electromagnetic wave in space. This prediction was demonstrated in 1888 by Hertz.

2. Maxwell's Equations

For the time-varying electromagnetic field, Maxwell summarized the following four equations:

The integral form	The differential form
$\oint_l \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \int_S (\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}) \cdot \mathrm{d}\boldsymbol{S}$	$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$
$\oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S}$	$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$
$\oint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0$	$\nabla \cdot \boldsymbol{B} = 0$
$\oint_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = q$	$\nabla \cdot \boldsymbol{D} = \rho$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad \nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \cdot \boldsymbol{D} = \rho$$

The time-varying electric field is both divergent and curly, and the time-varying magnetic field is solenoidal and curly. Nevertheless, the time-varying electric field and the time-varying magnetic field cannot be separated, and the time-varying electromagnetic field is divergent and curly.

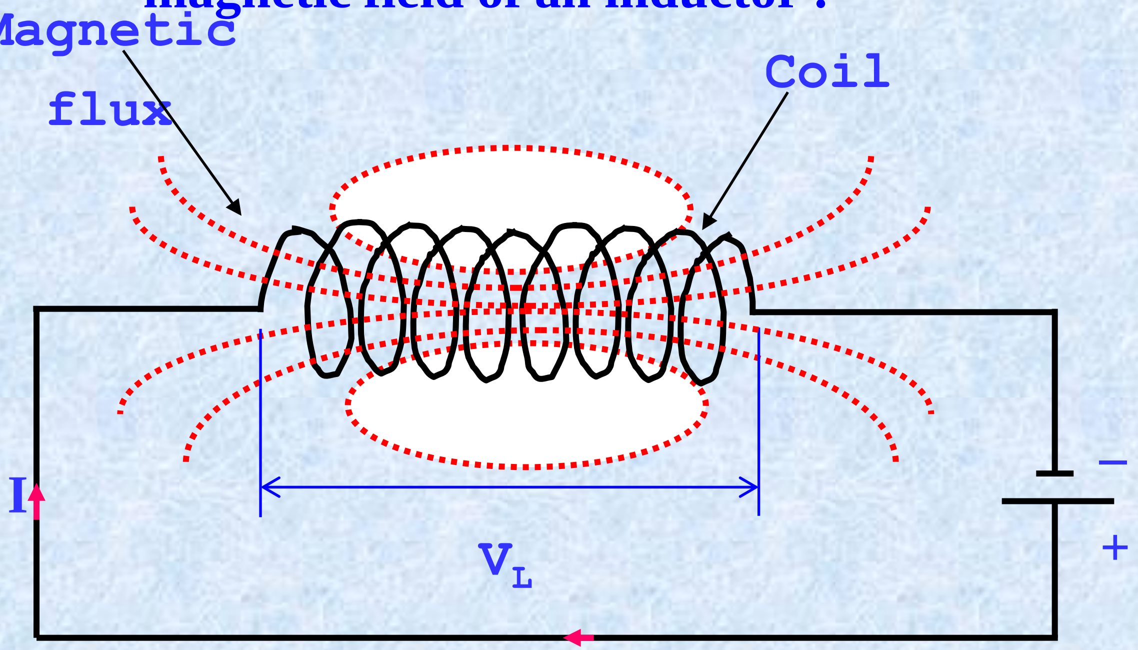
In a source-free region, the time-varying electromagnetic field is solenoidal.

The electric field lines and the magnetic field lines are linked with each other, forming closed loops, and resulting in an electromagnetic wave in space.

The time-varying electric field and the time-varying magnetic field are perpendicular to each other.

SELF INDUCTANCE AND MUTUAL INDUCTANCE

Simple electric circuit that shows the effect of energy stored in a magnetic field of an inductor :



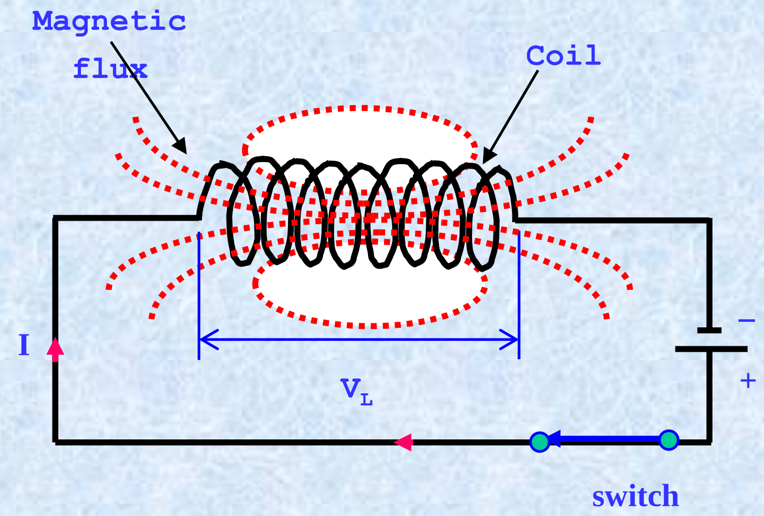
From circuit theory the induced potential across a wire wound coil such as solenoid or a toroid :

$$V_L = L\frac{dI}{dt}$$

where L is the inductance of the coil, I is the time varying current flowing through the coil – inductor.

In a capacitor, the energy is stored in the electric field : $W_E = \frac{1}{2}CV^2$

In an inductor, the energy is stored in the electric field, as suggested in the diagram :



$$W_m = \int_{t=0}^{t=t_0} V_L I dt = \int_{t=0}^{t=t_0} \left(L \frac{dI}{dt} \right) I dt$$

$$= \int_{t=0}^{t=t_0} LI\, dI = \frac{1}{2} LI^2 \quad (Joule)$$

Define the inductance of an inductor : $L \cong \frac{\Lambda}{I} \quad Henry$

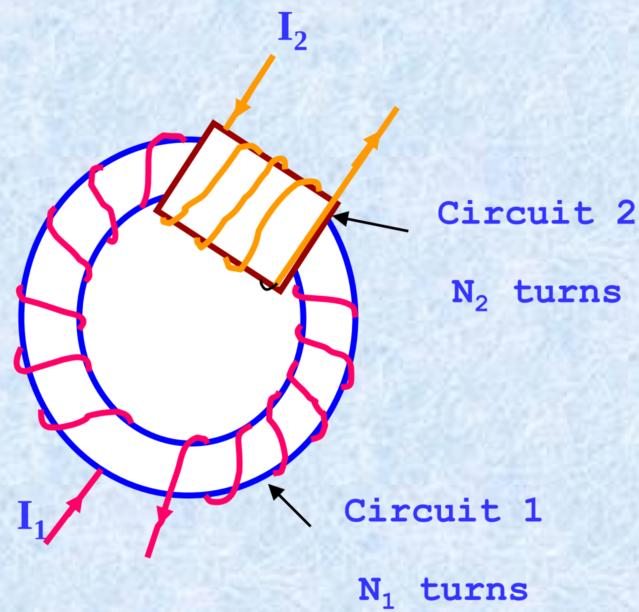
where Λ (lambda) is the total flux linkage of the inductor

$$L \cong \frac{\Lambda}{I} \quad H\ (Henry)$$

$$\Lambda = \psi_m N$$ **Weber turns**

Hence :

$$L = \frac{\psi_m N}{I}$$ **H**



Two circuits coupled by a common magnetic flux that leads to mutual inductance.

Mutual inductance :

$$M_{12} \cong \frac{\Lambda_{12}}{I_1}$$

Λ_{12} **is the linkage of circuit 2 produced by I_1 in circuit 1**

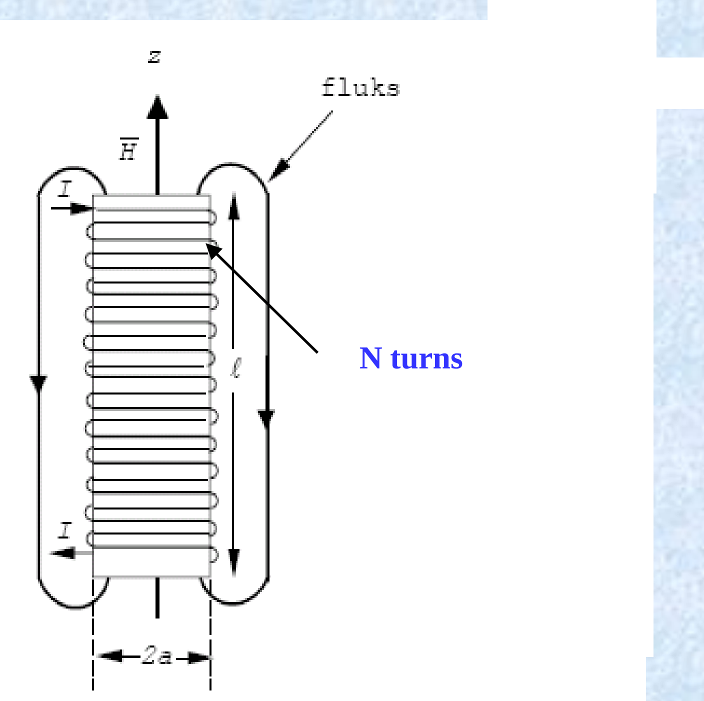
For linear magnetic medium

$$M_{12} = M_{21}$$

Obtain the self inductance of the long solenoid shown in the diagram.

Solution:

Assume all the flux ψ_m links all N turns and that $\overline{B}$ does not vary over the cross section area of the solenoid.



$$\Lambda = \psi_m N = B\left(\pi a^2\right)N$$

$$\text{We have } \overline{B} = \mu\overline{H}$$

$$\Lambda = (\mu H)\left(\pi a^2\right)N = \left(\frac{\mu NI}{l}\right)\left(\pi a^2\right)N$$

$$= \left(\frac{\mu N^2 I}{l}\right)\left(\pi a^2\right)$$

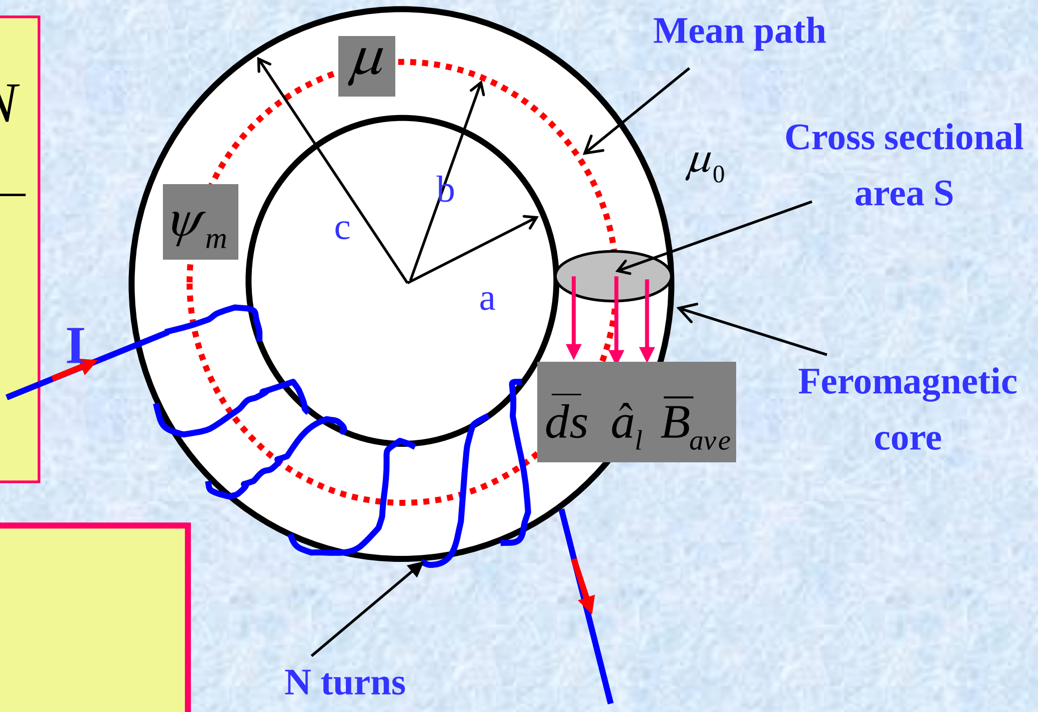
$$\therefore L = \frac{\Lambda}{I} = \frac{\mu N^2 \pi a^2}{l}$$

Obtain the self inductance of the toroid shown in the diagram.

Solution:

$$L = \frac{\Lambda}{I} = \frac{\psi_m N}{I} = \frac{B\left(\frac{\pi (c-a)^2}{4}\right)N}{I}$$

$$= \frac{\mu \frac{NI}{2\pi b} SN}{I}$$



$$\therefore L = \frac{\mu N^2 S}{2\pi b}$$

where b - mean radius

S - toroidal cross sectional area

Obtain the expression for self inductance per meter of the coaxial cable when the current flow is restricted to the surface of the inner conductor and the inner surface of the outer conductor as shown in the diagram.

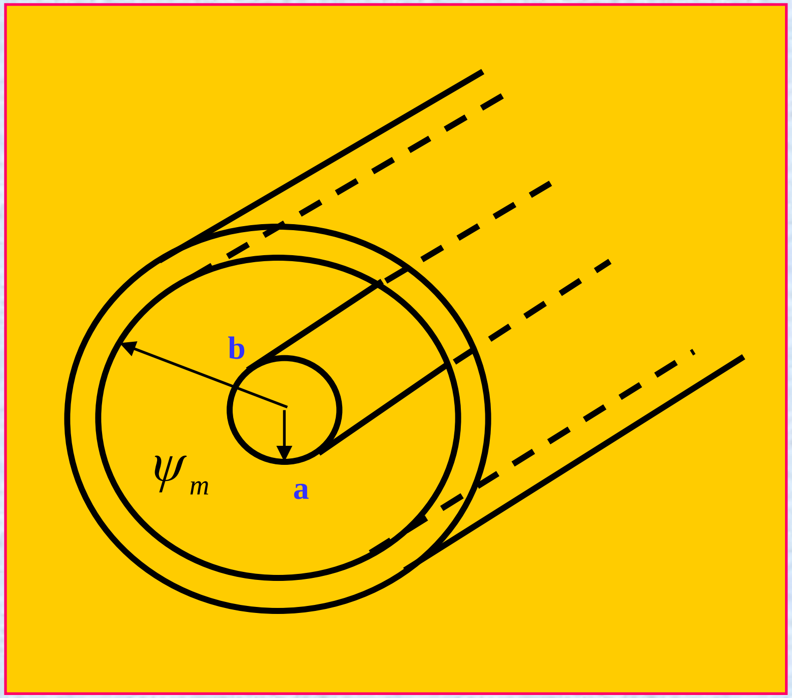
Solution:

The ψ_m **will exist only between a and b and will link all the current I**

$$L = \frac{\Lambda}{I} = \frac{\psi_m}{I} = \int_0^1 \int_a^b \frac{(\mu H)(dr_c dz)}{I}$$

$$= \int_0^1 \int_a^b \frac{\mu I}{2\pi r_c} \frac{(dr_c dz)}{I}$$

$$= \frac{\mu}{2\pi} \ln \frac{b}{a}$$



Find the expression for the mutual inductance between circuit 1 and circuit 2 as shown in the diagram.

Solution:

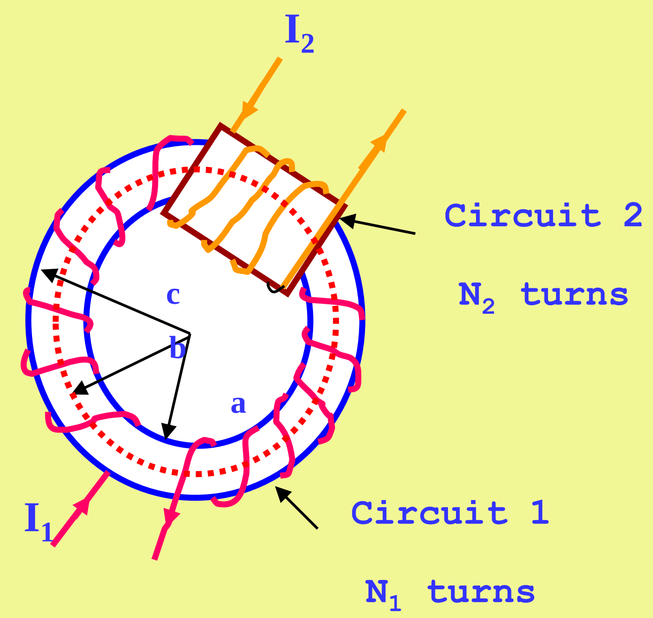
Let us assume the mean path :

$$2\pi b \gg (c-a)$$

$$M_{12} = \frac{\Lambda_{12}}{I_1} = \frac{\psi_{m(12)} N_2}{I_1}$$

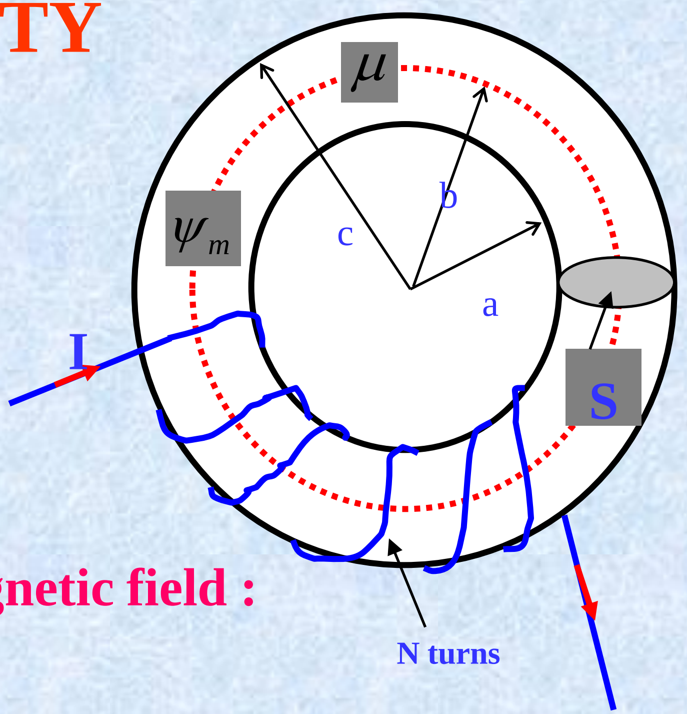
$$= \frac{B_{12}\left(\frac{\pi (c-a)^2}{4}\right) N_2}{I_1}$$

$$= \frac{\mu \frac{N_1 I_1}{2\pi b} S N_2}{I_1} = \frac{\mu N_1 N_2 S}{2\pi b}$$



Two circuits coupled by a common magnetic flux that leads to mutual inductance.

MAGNETIC ENERGY DENSITY



We have :

$$L \cong \frac{\Lambda}{I} \quad \text{Henry}$$

$$W_m = \frac{1}{2} L I^2 = \frac{1}{2} \frac{\Lambda}{I} I^2 = \frac{1}{2} \Lambda I \quad \text{Joule}$$

Consider a toroidal ring : The energy in the magnetic field :

$$W_m = \frac{1}{2} \psi_m N I = \frac{1}{2} B S N I$$

Multiplying the numerator and denominator by 2πb :

$$W_m = \frac{1}{2} B \frac{NI}{2\pi b} \left(S 2\pi b \right)$$

where $\frac{NI}{2\pi b} = H$ and $(S 2\pi b)$ is the volume V of the toroid

Hence :

$$W_m = \frac{1}{2} BHV$$

$$w_m = \frac{W_m}{V} = \frac{1}{2} BH = \frac{1}{2} \mu H^2 \qquad Jm^{-3}$$

In vector form :

$$w_m = \frac{1}{2} \overline{B} \cdot \overline{H}$$

Hence the inductance :

$$L = \frac{2}{I^2} W_m = \frac{2}{I^2} \int_v \frac{1}{2} \overline{B} \cdot \overline{H} \, dv$$

Derive the expression for stored magnetic energy density in a coaxial cable with the length l and the radius of the inner conductor a and the inner radius of the outer conductor is b. The permeability of the dielectric is μ .

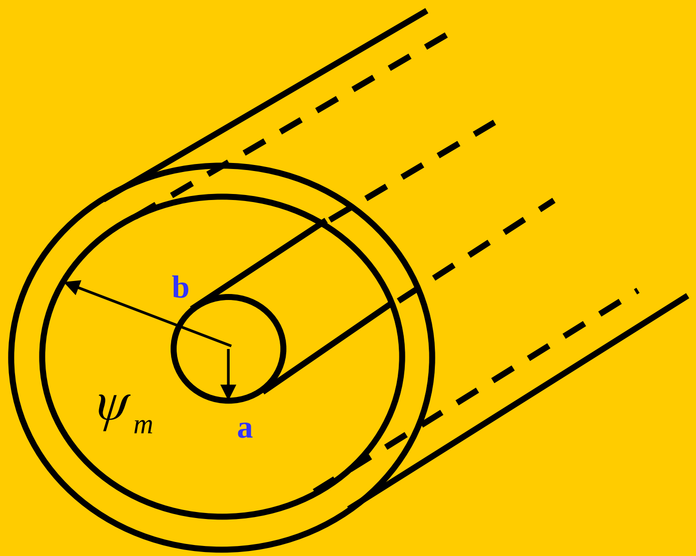
Solution:

$$H = \frac{I}{2\pi r}$$

$$W_m = \frac{1}{2}\int_v \mu H^2 dv = \frac{\mu I^2}{8\pi^2}\int_v \frac{1}{r^2}dv$$

$$W_m = \frac{\mu I^2}{8\pi^2}\int_a^b \frac{1}{r^2}\cdot 2\pi r l\, dr$$

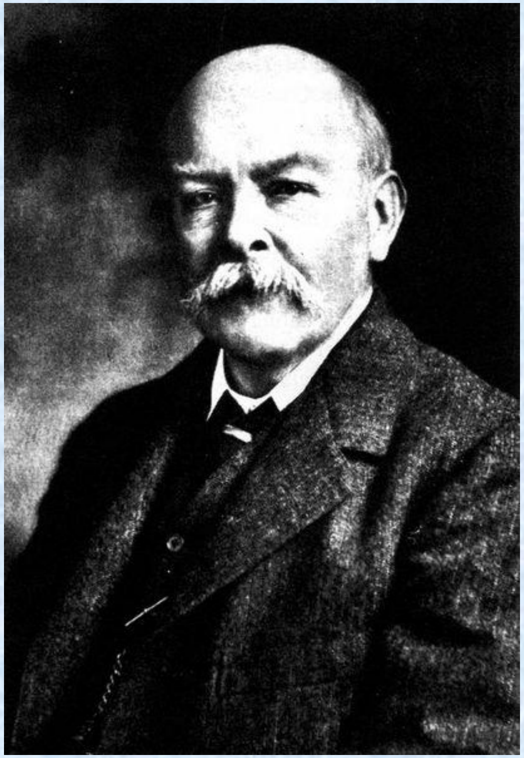
$$= \frac{\mu I^2 l}{4\pi}\ln\left(\frac{b}{a}\right) \quad \text{(J)}$$



Poynting Theorem

The Poynting theorem is one of the most important results in EM theory. It tells us the power flowing in an electromagnetic field.

John Henry Poynting (1852-1914)



John Henry Poynting was an English physicist. He was a professor of physics at Mason Science College (now the University of Birmingham) from 1880 until his death.

He was the developer and eponym of the Poynting vector, which describes the direction and magnitude of electromagnetic energy flow and is used in the Poynting theorem, a statement about energy conservation for electric and magnetic fields. This work was first published in 1884. He performed a measurement of Newton's gravitational constant by innovative means during 1893.

Poynting Theorem

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}$$

From these we obtain

$$\underline{H} \cdot \left(\nabla \times \underline{E}\right) = -\underline{H} \cdot \frac{\partial \underline{B}}{\partial t}$$

$$\underline{E} \cdot \left(\nabla \times \underline{H}\right) = \underline{J} \cdot \underline{E} + \underline{E} \cdot \frac{\partial \underline{D}}{\partial t}$$

Poynting Theorem (cont.)

$$\underline{H} \cdot \left(\nabla \times \underline{E} \right) = -\underline{H} \cdot \frac{\partial \underline{B}}{\partial t}$$

$$\underline{E} \cdot \left(\nabla \times \underline{H} \right) = \underline{J} \cdot \underline{E} + \underline{E} \cdot \frac{\partial \underline{D}}{\partial t}$$

Subtract, and use the following vector identity:

$$\underline{H} \cdot \left(\nabla \times \underline{E} \right) - \underline{E} \cdot \left(\nabla \times \underline{H} \right) = \nabla \cdot \left(\underline{E} \times \underline{H} \right)$$

We then have:

$$\nabla \cdot \left(\underline{E} \times \underline{H} \right) = -\underline{J} \cdot \underline{E} - \underline{H} \cdot \frac{\partial \underline{B}}{\partial t} - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t}$$

Poynting Theorem (cont.)

$$\nabla \cdot \left(\underline{E} \times \underline{H} \right) = -\underline{J} \cdot \underline{E} - \underline{H} \cdot \frac{\partial \underline{B}}{\partial t} - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t}$$

Next, assume that Ohm's law applies for the electric current:

$$\underline{J} = \sigma \, \underline{E}$$

$$\Rightarrow \quad \nabla \cdot \left(\underline{E} \times \underline{H} \right) = -\sigma \left(\underline{E} \cdot \underline{E} \right) - \underline{H} \cdot \frac{\partial \underline{B}}{\partial t} - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t}$$

or

$$\nabla \cdot \left(\underline{E} \times \underline{H} \right) = -\sigma \left| \underline{E} \right|^2 - \underline{H} \cdot \frac{\partial \underline{B}}{\partial t} - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t}$$

Poynting Theorem (cont.)

$$\nabla \cdot \left(\underline{E} \times \underline{H} \right) = -\sigma \left| \underline{E} \right|^2 - \underline{H} \cdot \frac{\partial \underline{B}}{\partial t} - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t}$$

From calculus (chain rule), we have that

$$\underline{E} \cdot \frac{\partial \underline{D}}{\partial t} = \varepsilon \left(\underline{E} \cdot \frac{\partial \underline{E}}{\partial t} \right) = \varepsilon \frac{1}{2} \frac{\partial}{\partial t} \left(\underline{E} \cdot \underline{E} \right)$$

$$\underline{H} \cdot \frac{\partial \underline{B}}{\partial t} = \mu \left(\underline{H} \cdot \frac{\partial \underline{H}}{\partial t} \right) = \mu \frac{1}{2} \frac{\partial}{\partial t} \left(\underline{H} \cdot \underline{H} \right)$$

Hence we have

$$\nabla \cdot \left(\underline{E} \times \underline{H} \right) = -\sigma \left| \underline{E} \right|^2 - \mu \frac{1}{2} \frac{\partial}{\partial t} \left(\underline{H} \cdot \underline{H} \right) - \varepsilon \frac{1}{2} \frac{\partial}{\partial t} \left(\underline{E} \cdot \underline{E} \right)$$

Poynting Theorem (cont.)

$$\nabla \cdot \left(\underline{E} \times \underline{H} \right) = -\sigma \left| \underline{E} \right|^2 - \mu \frac{1}{2} \frac{\partial}{\partial t} \left(\underline{H} \cdot \underline{H} \right) - \varepsilon \frac{1}{2} \frac{\partial}{\partial t} \left(\underline{E} \cdot \underline{E} \right)$$

This may be written as

$$\nabla \cdot \left(\underline{E} \times \underline{H} \right) = -\sigma \left| \underline{E} \right|^2 - \mu \frac{1}{2} \frac{\partial}{\partial t} \left| \underline{H} \right|^2 - \varepsilon \frac{1}{2} \frac{\partial}{\partial t} \left| \underline{E} \right|^2$$

or

$$\nabla \cdot \left(\underline{E} \times \underline{H} \right) = -\sigma \left| \underline{E} \right|^2 - \frac{\partial}{\partial t} \left(\frac{1}{2} \mu \left| \underline{H} \right|^2 \right) - \frac{\partial}{\partial t} \left(\frac{1}{2} \varepsilon \left| \underline{E} \right|^2 \right)$$

Poynting Theorem (cont.)

Final differential form of the Poynting theorem:

$$\nabla \cdot \left(\underline{E} \times \underline{H} \right) = -\sigma \left| \underline{E} \right|^2 - \frac{\partial}{\partial t}\left(\frac{1}{2} \mu \left| \underline{H} \right|^2 \right) - \frac{\partial}{\partial t}\left(\frac{1}{2} \varepsilon \left| \underline{E} \right|^2 \right)$$

Poynting Theorem (cont.)

Volume (integral) form

Integrate both sides over a volume and then apply the divergence theorem:

$$\nabla \cdot \left(\underline{E} \times \underline{H} \right) = -\sigma \left| \underline{E} \right|^2 - \frac{\partial}{\partial t}\left(\frac{1}{2}\mu \left| \underline{H} \right|^2 \right) - \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon \left| \underline{E} \right|^2 \right)$$

$$\int_V \nabla \cdot \left(\underline{E} \times \underline{H} \right) dV = -\int_V \sigma \left| \underline{E} \right|^2 dV - \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\mu \left| \underline{H} \right|^2 \right) dV - \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon \left| \underline{E} \right|^2 \right) dV$$

$$\oint_S \left(\underline{E} \times \underline{H} \right) \cdot \hat{n}\, dS = -\int_V \sigma \left| \underline{E} \right|^2 dV - \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\mu \left| \underline{H} \right|^2 \right) dV - \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon \left| \underline{E} \right|^2 \right) dV$$

Poynting Theorem (cont.)

Final volume form of Poynting theorem:

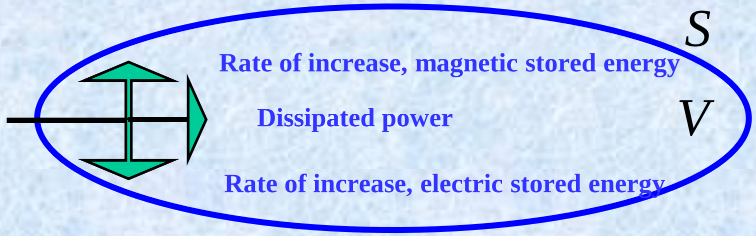
$$\oint_S \left(\underline{E}\times\underline{H}\right)\cdot\hat{n}\,dS = -\int_V \sigma\left|\underline{E}\right|^2 dV - \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\mu\left|\underline{H}\right|^2\right)dV - \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon\left|\underline{E}\right|^2\right)dV$$

For a stationary surface:

$$\oint_S \left(\underline{E}\times\underline{H}\right)\cdot\hat{n}\,dS = -\int_V \sigma\left|\underline{E}\right|^2 dV - \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\mu\left|\underline{H}\right|^2\right)dV - \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\varepsilon\left|\underline{E}\right|^2\right)dV$$

Poynting Theorem (cont.)

Power in



Physical interpretation:

(Assume that S is stationary.)

$$-\oint_S \left(\underline{E}\times\underline{H}\right)\cdot\hat{n}\,dS = \int_V \sigma\left|\underline{E}\right|^2 dV + \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\mu\left|\underline{H}\right|^2\right) dV + \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\varepsilon\left|\underline{E}\right|^2\right) dV$$

Power dissipation as heat (Joule's law)

Rate of change of stored magnetic energy

Rate of change of stored electric energy

$\Longrightarrow$ Right-hand side = power flowing into the volume of space.

Poynting Theorem (cont.)

Hence

$$-\oint_S \left(\underline{E}\times\underline{H}\right)\cdot\hat{n}\,dS = \text{power flowing into the region}$$

Or, we can say that

$$\oint_S \left(\underline{E}\times\underline{H}\right)\cdot\hat{n}\,dS = \text{power flowing } \textit{out} \text{ of the region}$$

Define the Poynting vector: $\underline{S} = \underline{E}\times\underline{H}$

Poynting Theorem (cont.)

Analogy:

$$\oint_S \underline{S} \cdot \hat{n}\, dS = \text{power flowing out of the region}$$

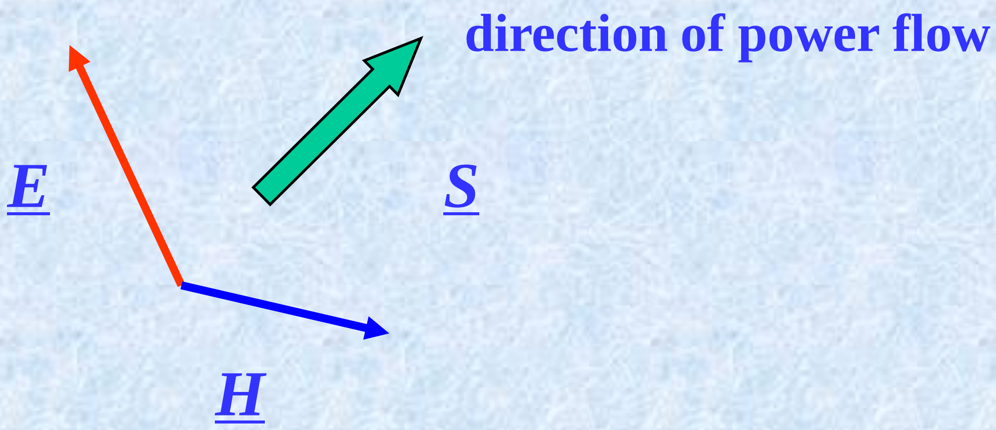
$$\oint_S \underline{J} \cdot \hat{n}\, dS = \text{current flowing out of the region}$$

$\underline{J}$ = current density vector

$\underline{S}$ = power flow vector

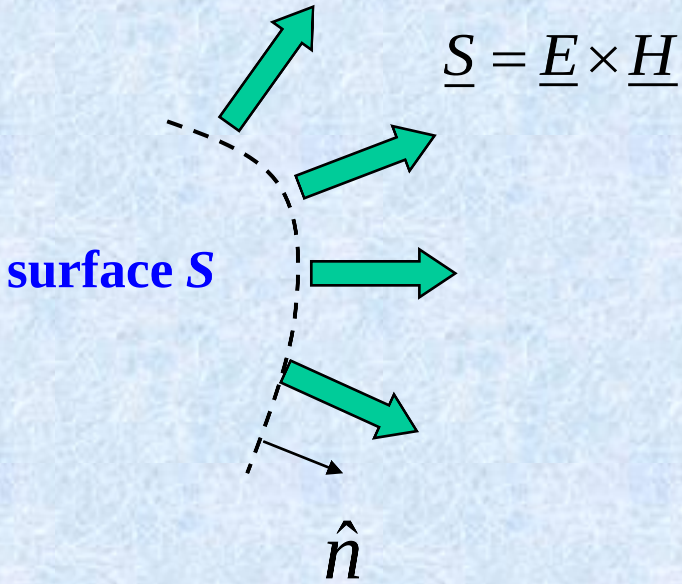
Poynting Theorem (cont.)

$$\underline{S} = \underline{E} \times \underline{H}$$



The units of $\underline{S}$ are [W/m^2].

Power Flow



The power *P* flowing through the surface *S* (from left to right) is:

$$P = \int_S \underline{S} \cdot \hat{n}\, dS$$

Time-Average Poynting Vector

Assume sinusoidal (time-harmonic) fields)

$$\underline{S}(x,y,z,t) = \underline{E}(x,y,z,t) \times \underline{H}(x,y,z,t)$$

$$\underline{E}(x,y,z,t) = \text{Re}\left\{\underline{\text{E}}(x,y,z)e^{j\omega t}\right\}$$

$$\underline{H}(x,y,z,t) = \text{Re}\left\{\underline{\text{H}}(x,y,z)e^{j\omega t}\right\}$$

From our previous discussion (notes 2) about time averages, we know that

$$\left\langle \underline{S}(t)\right\rangle = \left\langle \underline{E}(t) \times \underline{H}(t)\right\rangle = \frac{1}{2}\text{Re}\left(\underline{\text{E}} \times \underline{\text{H}}^*\right)$$

Complex Poynting Vector

Define the complex Poynting vector:

$$\underline{S} \equiv \frac{1}{2}\left(\underline{E} \times \underline{H}^{*}\right)$$

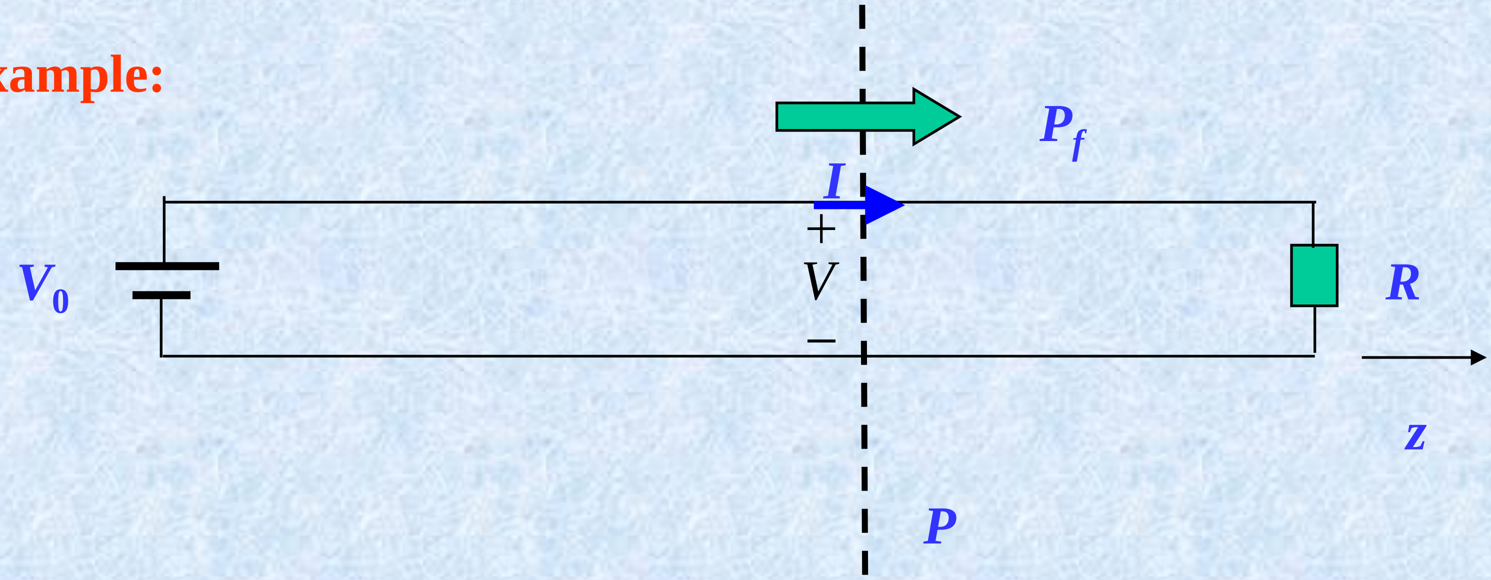
We then have that

$$\left\langle \underline{S}\left(x, y, z, t\right)\right\rangle = \mathrm{Re}\left(\underline{S}\left(x, y, z\right)\right)$$

Note on Circuit Theory

Although the Poynting vector can always be used to calculate power flow, at low frequency, circuit theory can be used, and this is usually easier.

Example:



$$P_f = \int_P \left(\underline{E} \times \underline{H} \right) \cdot \underline{\hat{z}}\, dS$$

$$P_f = V\, I$$

or, in frequency domain

$$\left\langle P_f \right\rangle = \frac{1}{2} \mathrm{Re} \int_P \left(\underline{E} \times \underline{H}^* \right) \cdot \underline{\hat{z}}\, dS$$

$$\left\langle P_f \right\rangle = \frac{1}{2} \mathrm{Re} \left(V\, I^* \right)$$